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Analytical solution of the string vibration model on Sasando musical instrument

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Abstract. This research constructs mathematic model for string vibration on Sasando that expressed as a second-order linear partial differential equation with u denote the vertical displacement experienced by the string at the point x at time t . Besides, the purpose of this research is to determine analytical solution using a method known as separation of variable. There are three analytical solutions from three cases based on constant coefficient values. Case 1 occurs when $k_d^2 > 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$. Based on he simulation, the string produces a quarter wave within 3 seconds. This means the vibration wavelength 0.083 Hz. In this case the vertical displacement of the string is back to quiescent after vibrating to produce a quarter wave. Case 2 occurs when $k_d^2 = 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$. The string produces 3.5 waves within 3 seconds, means the vibration wavelength is 1.67 Hz. In this case, the farthest vertical displacement (amplitude) become smaller drastically and back to quiescent. Case 3 occurs when $k_d^2 < 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$. The vibration wavelength of the string is 2.83 Hz, with 7.5 waves within 3 seconds. The vertical displacement decreases periodically and back to quiescent at a certain time. Analysis of 2-dimensional graph for vertical displacement $u(x, t)$ is done with $n = 1, 2, 3, 4, 5$ which is compared with profile of $u(x, t)$ Gulla (2011). Based on results of the model were mentioned before, the mathematical model of string vibration on Sasando has the similar profile as Gulla (2011). For further, this research is suggested to perform a graph profile analysis of other cases that may arise with variation of parameter values.

1. Introduction

This research focused to constructs mathematical model of string vibration on Sasando. String vibration included the torque motion and it's important to analyzed deflection behavior [1]. Lagrange equation were used to construct the model. The mathematical model of string vibration on Sasando is a two-order partial differential equation with the dependent variable u showing the vertical displacement at point x and at time t . Before it is implemented, the model must be validated. Validation test is performed to verify the validity of the model. a valid model is a model that provides an interpretation of the results approaching the real system. If the model does not satisfy the validation requirements, the model must be repaired and reformulated.



A model about string vibration on electric guitar has been constructed in the literature [2] and [3]. The mathematical analysis on [2] depend on calculation of guitar strings frequency based on octave relationship of the frets and physical parameters such as linear mass density and length of the string.

Validation test sometimes done by simulation to display graph profile without analyzing the graph profile. This validation test is not enough to show the validity of the model. Therefore, it is necessary to perform an advanced validation test to complete the model validation effort. The model validation test conducted in this study is to determine the analytic solution of the model. The urgency of the analytic solution as a validator is to find out the vertical displacement u at point x and time t . Based on the results of graph profile analysis, researchers can determine the validity of the model

The analytical solution of the string vibration model on Sasando was obtained by mathematical calculation using separation of variable method. The main point of this method is the replacement of partial differential equations with a set of ordinary differential equations. The required solution of the partial differential equation is expressed as the product of ordinary differential equations solution. Furthermore, to obtain a solution, this study refers to the initial value $f(x)$ in the case of acoustic guitar vibrations on [3]. The solution is simulated to get the graph profile. Graph profiles are analyzed to determine the validity of the model.

Based on the above explanation, this research will be focused on constructing and determining the solution of string vibration model on Sasando and analyzing the results of the simulation in order to test the validation of string vibration model on Sasando.

2. Methods

Construction done by using Lagrange equation and analytical solutions are obtained by applying the so-called method of separating variables or *product method*, obtained two ordinary differential equation. The solutions determined of those two equations that satisfy the boundary conditions. Those solutions composed using *Fourier* series in order to get a solution of string vibration model on Sasando.

3. Results and Discussion

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{1}{2}c^2 + 2\frac{c^2}{l} \right) \frac{\partial^2 u}{\partial x^2} + k_a \frac{\partial u}{\partial t} = 0 \quad (1)$$

The equation involves partial derivatives of any function consisting of two or more variables are called partial differential equation (PDE) [4]. Based on the form, the equation called one dimensional wave equation [5]. It has only one space dimension as x and one independent variable t . With the initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x, 0) = 0 \quad (2)$$

The initial value of $f(x)$ is defined as follows

$$f(x) = \begin{cases} \frac{hx}{d}, & 0 \leq x \leq d \\ \frac{h(l-x)}{l-d}, & d \leq x \leq l \end{cases} \quad (3)$$

and with boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(l, t) = 0 \quad (4)$$

Based on (4) the following boundary conditions are found

$$X(0) \quad \text{and} \quad X(l) = 0 \quad (5)$$

The following is form of solution of wave equation [6].

$$u(x, t) = X(x)T(t)$$

Using separation of variable method, the string vibration model on Sasando becomes two ODEs with each one independent variable point along the string x and time t as follows:

$$\frac{X''(x)}{X(x)} = -k \quad (6)$$

and

$$\frac{1}{\left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)}\frac{\ddot{T}(t)}{T(t)} + \frac{k_d}{\left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)}\frac{\dot{T}(t)}{T(t)} = -k \quad (7)$$

Where k is the separating constant. Based on [7], determined of two equations that satisfy the boundary conditions.

a. Solution of equation (6)

$$X''(x) + kX(x) = 0$$

Case I; $\forall k < 0$ let $k = -\lambda^2$ where λ real. So the equation (6) becomes $X''(x) - \lambda^2 X(x) = 0$ and the general solution is

$$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

Boundary condition (5) indicate that $C_1 = -C_2$ with $C_2 = 0$ or $C_2 \neq 0$. For $C_2 = 0$, then $C_1 = 0$ so there is no nontrivial solution. Next to $C_2 \neq 0$, then $(-e^{\lambda l} + e^{-\lambda l}) = 0$ and obtained $2\lambda l = 0 \forall \lambda, l \neq 0$. So in this case there is no nontrivial solution.

Case II; $k = 0$, the equation

(6) becomes $X''(x) = 0$ and the general solution is

$$X(x) = C_1 + xC_2$$

the boundary condition (5) indicate that $C_1 = C_2 = 0$. So there is no nontrivial solution.

Case III, $\forall k > 0$ let $k = \lambda^2$ where λ real. So the equation (3.6) becomes $X''(x) + \lambda^2 X(x) = 0$ and the general solution is

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

boundary condition (5) indicate that $C_1 = 0$ and $\sin \lambda l = 0$ ie $\lambda = \frac{n\pi}{l}$, where $n = 1, 2, 3, \dots$, so that a solution is obtained

$$X(x) = C_2 \sin\left(\frac{n\pi}{l} x\right) \quad (8)$$

b. Solution of equation (7)

Remember

$\lambda = \frac{n\pi}{l}$, $n = 1, 2, 3, \dots$, with $k = \lambda^2$ so $k = \left(\frac{n\pi}{l}\right)^2$, and equation (3.7) becomes

$$\ddot{T}(t) + k_d \dot{T}(t) + \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right) T(t) = 0$$

Case 1; if $k_d^2 > 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$, then the general solution is

$$T_n(t) = C_3 e^{\frac{-k_d + \sqrt{k_d^2 - 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)}}{2} t} + C_4 e^{\frac{-k_d - \sqrt{k_d^2 - 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)}}{2} t}$$

Suppose that f is a function defined on the interval $[-p, p]$. The Fourier series of an even function on the interval is the cosine series [8]. Using Fourier series is a solution to wave equation [9]. The solution is composed to get a solution that satisfy initial or boundary condition using Fourier Series [10]. Then using initial condition and the definition of Fourier sine series, obtained

$$A_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

with $f(x)$ as defined in ((3), A_n can be known using partial integral.

$$A_n = \left(\frac{2hl^2}{n^2\pi^2 d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right) \quad (9)$$

Furthermore, using the second boundary condition B_n known as follows:

$$B_n = \frac{k_d \left(\frac{2hl^2}{n^2\pi^2 d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right)}{\sqrt{k_d^2 - 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)}} \quad (10)$$

Case 2; if $k_d^2 = 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$, then the general solution is

$$T_n(t) = C_3 e^{-\frac{k_d}{2}t} + C_4 e^{-\frac{k_d}{2}t} \quad (11)$$

Then using initial condition and the definition of Fourier sine series, obtained

$$f(x) = A_n \sin\left(\frac{n\pi}{l}x\right)$$

With $f(x)$ as defined in ((3), A_n can be known using partial integral

$$A_n = \left(\frac{2hl^2}{n^2\pi^2 d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right) \quad (11)$$

Furthermore, using the second boundary condition B_n known as follows:

$$B_n = \frac{k_d}{2} \left(\frac{2hl^2}{n^2\pi^2 d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right) \quad (12)$$

Case 3; if $k_d^2 < 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$, then the general solution is

$$T_n(t) = C_3 e^{\frac{-k_d + i \sqrt{\left|k_d^2 - 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)\right|}}{2} t} + C_4 e^{\frac{-k_d - i \sqrt{\left|k_d^2 - 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)\right|}}{2} t} \quad (14)$$

Then using initial condition and the definition of Fourier sine series, obtained

$$f(x) = a_n \sin\left(\frac{n\pi}{l}x\right)$$

With $f(x)$ as defined in ((3), a_n can be known using partial integral.

$$a_n = \left(\frac{2hl^2}{n^2\pi^2d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right) \quad (13)$$

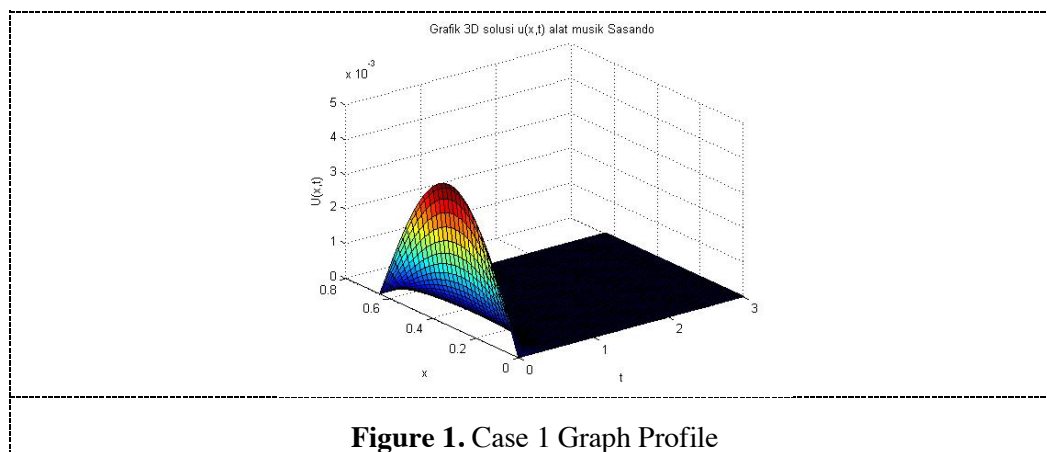
Next, using the second boundary condition b_n known as follows:

$$b_n = \frac{\left(\left(\frac{2hl^2}{n^2\pi^2d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right)\right) k_d}{\sqrt{\left|k_d^2 - 4\left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)\right|}} \quad (14)$$

Similar with step on [11], substituting (9) and (10), (11) and (12), (13) and (14) into the solution series, we have solutions of case 1, case 2 and case 3.

Case 1

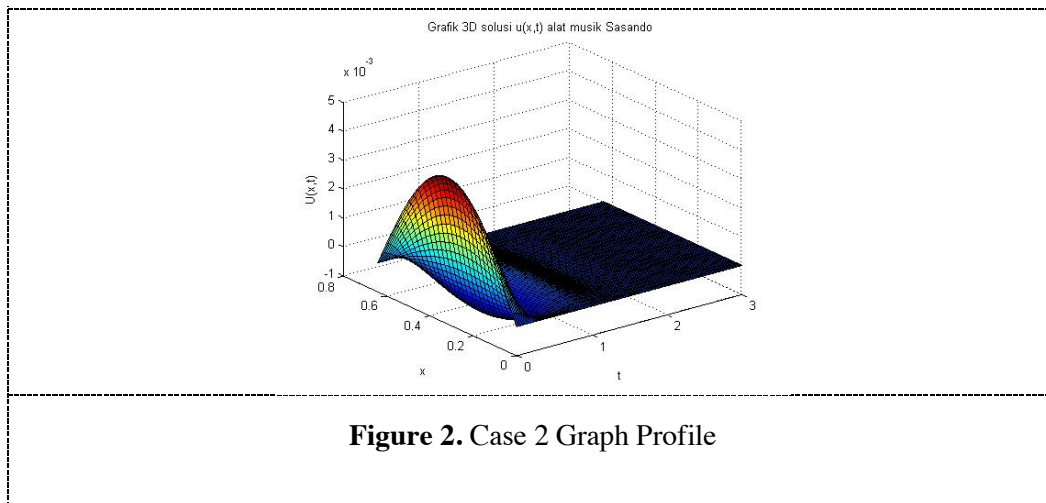
Selected parameter value $k_d = 29.5$, $c = 1$ and $l = 0.64$. The following is a graph of the MATLAB-generated solution of case 1 with the selected parameter value as shown in Figure 1.



From the result of the graph profile of case 1 solution, within the first 3 seconds the string produces a quarter wave i.e. one peak and does not experience vibration again afterwards so that the vibration frequency of the strings is very small which is about 0.083 Hz.

Case 2

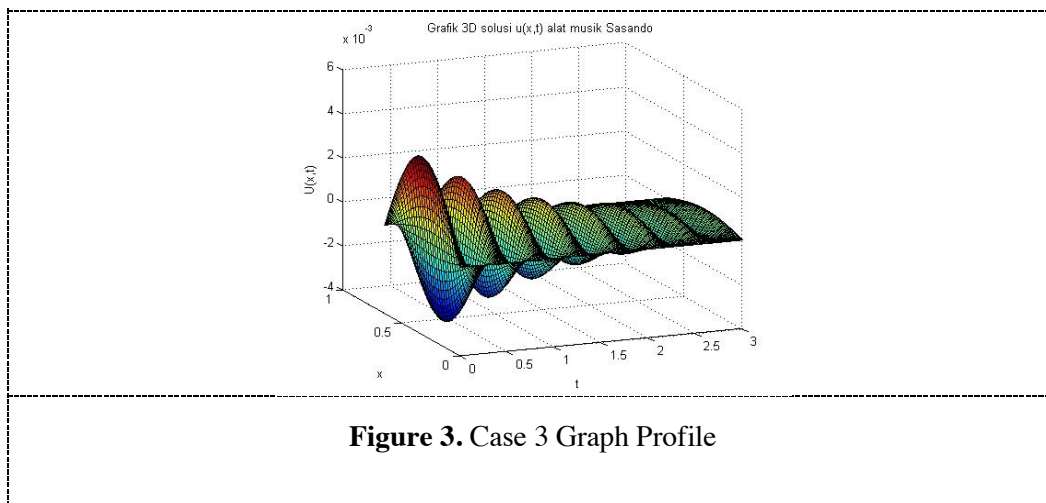
Selected parameter value $k_d = 6.283185310$, $c = 0.5$ and $l = 0.64$. The following is a graph of the MATLAB-generated solution of case 2 with the selected parameter values.



From the results of the graph profile of the case 2 solution, in the first 3 seconds the string produced 3.5 waves, namely 4 peaks and 3 valleys. This means that the vibration frequency of the strings is 1.67 Hz. In addition, the amplitude is also very small which is around 10^{-25} , as shown in Figure 2.

Case 3

The parameter values $k_d = 1.5$, $c = 1$ and $l = 0.64$. The following is a graph of the solution generated by MATLAB from case 3 with the selected parameter value.



From the results of the graph case in Figure 3, the amplitude of string vibration on Sasando is smaller as time t goes until it finally stops or back to quiescent. The results of this visualization illustrate the real system of the strings vibration on Sasando. The string of Sasando vibrates temporarily and stop after a few seconds.

Judging from the number of waves, in the first 3 seconds Sasando string produced 7.5 waves, namely 8 peaks and 7 valleys, means the frequency of the string vibration is 2.83 Hz. From the 3 existing cases, the frequency generated from case 3 is the largest frequency.

The value of parameters just chosen, and possible to any value. The value can be chosen by assumption as in literature [9]. The researchers mean string is flexible and has no internal losses and damping so the researchers choose certain parameter value.

Next presented the graph simulation results of string vibration solutions as a reference with $c = 4$, $l = 0.64$ and $d = 0.16$ as shown in Figure 4

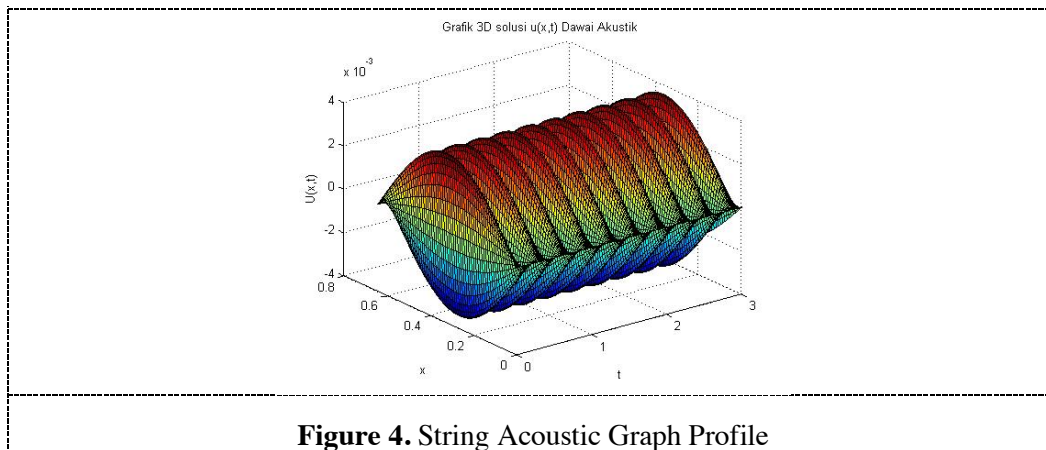


Figure 4. String Acoustic Graph Profile

Judging from the number of waves, in the first 3 seconds the string produces 9.5 waves, namely 10 peaks and 9 valleys, means that the string vibration frequency is 3.16 Hz.

Furthermore, the researcher will use a difference value of the amplitude of Sasando string vibration and acoustic string vibration as a consideration. Amplitude of the acoustic string with $d = 0.16$ are presented in Table 1.

Table 1. Amplitude for $d = 0.16$

n	Acoustic String Amplitude	Sasando Amplitude	Difference
1	3.821×10^{-3}	4.052×10^{-3}	0.231×10^{-3}
2	1.351×10^{-3}	1.519×10^{-35}	1.351×10^{-3}
3	4.246×10^{-4}	4.503×10^{-4}	0.257×10^{-4}
4	4.138×10^{-20}	1.519×10^{-35}	4.138×10^{-20}
5	-1.528×10^{-4}	1.621×10^{-4}	3.149×10^{-4}

The difference of amplitude on Table 1. Amplitude for $d = 0.16$

n	Acoustic String Amplitude	Sasando Amplitude	Difference
1	3.821×10^{-3}	4.052×10^{-3}	0.231×10^{-3}
2	1.351×10^{-3}	1.519×10^{-35}	1.351×10^{-3}
3	4.246×10^{-4}	4.503×10^{-4}	0.257×10^{-4}
4	4.138×10^{-20}	1.519×10^{-35}	4.138×10^{-20}
5	-1.528×10^{-4}	1.621×10^{-4}	3.149×10^{-4}

presented in Figure 5:

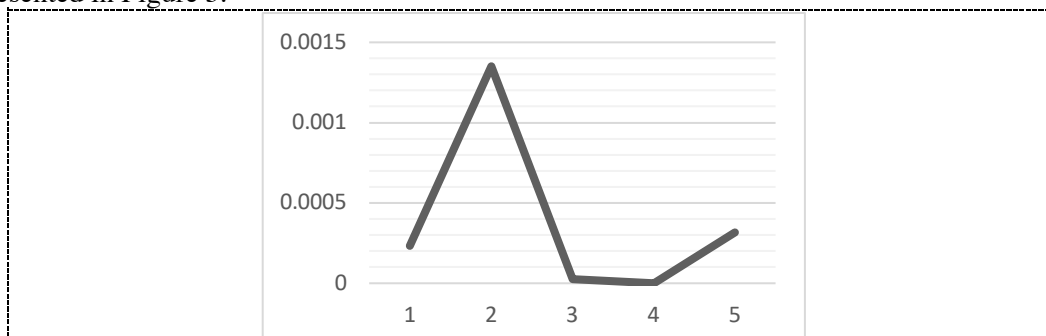


Figure 5. Difference of Amplitude

The biggest amplitude difference between Sasando and acoustic strings is at $n = 2$ and $n = 5$.

4. Conclusion

There are 3 solutions of string vibration model on Sasando. Each solution is obtained based on the case with different parameter. The solutions for each case are as follows:

Case 1; $k_d^2 > 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$

$$u(x, t) = \sum_{n=1}^{\infty} e^{-\frac{k_d}{2}t} \sin\left(\frac{n\pi}{l}x\right) \left(\frac{2hl^2}{n^2\pi^2d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right) \left(\cosh\left(\frac{\sqrt{k_d^2 - 4\left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)}}{2}t\right) + \frac{k_d}{\sqrt{k_d^2 - 4\left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)}} \sinh\left(\frac{\sqrt{k_d^2 - 4\left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)}}{2}t\right)\right)$$

Case 2; $k_d^2 = 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}x\right) \left(\frac{2hl^2}{n^2\pi^2d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right) e^{-\frac{k_d}{2}t} + \frac{k_d}{2} \left(\frac{2hl^2}{n^2\pi^2d(l-d)}\right) \sin\left(\frac{n\pi d}{l}\right) t e^{-\frac{k_d}{2}t}$$

Case 3; $k_d^2 < 4 \left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)$

$$u(x, t) = \sum_{n=1}^{\infty} \left(\frac{2hl^2}{n^2\pi^2d(l-d)}\right) \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{n\pi d}{l}\right) e^{-\frac{k_d}{2}t} \left(\cos\left(\frac{\sqrt{\left|k_d^2 - 4\left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)\right|}}{2}t\right) + \frac{k_d}{\sqrt{\left|k_d^2 - 4\left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)\right|}} \sin\left(\frac{\sqrt{\left|k_d^2 - 4\left(\frac{n\pi}{l}\right)^2 \left(\frac{1}{2}c^2 + 2\frac{c^2}{l}\right)\right|}}{2}t\right)\right)$$

References

- [1] Kusumastuti A and Nurhayati S 2016 *J. Tech.* **78** 351
- [2] Atre M P and Apte S 2017 *Int. J. Signal Process.* **2** 2
- [3] Gulla J 2012 *Modeling the Wave Motion of a Guitar String* (Swiss: Int. Baccalaureate®) p 10
- [4] O'Neil P V 2014 *Beginning Partial Differential Equations* (USA: John Wiley & Sons Inc)

- [5] Saiduzzaman M, Syaikh Y and Samrat S M 2014 *Global J. Sci. Front. Res.* **14** 33
- [6] Geetha H, Sudha T and Srinivas H 2017 *Int. J. Pure Appl. Math.* **117** 167
- [7] Coleman M P 2013 *An Introduction to Partial Differential Equations with MATLAB®* (Boca Raton: CRC Press)
- [8] Haberman R 2013 *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems* (Canada: Pearson)
- [9] Hammond J 2011 *UW-L J. Undergraduate Res.* **17** 1
- [10] Vera M R, Niño Y and Valencia A 2012 *Math. Prob. Eng.* **2012** 1
- [11] Akinlabi G and Edeki S 2016 *Int. J. Pure Appl. Math.* **107** 449