STUDY OF THE GRAVITY EFFECTS OF FERMION AND BOSON PARTICLES IN CURVED SPACETIME

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ABSTRACT

The types of particles used in this research are Fermion particles and Boson particles. So to describe the movement of Fermion and Boson particles, the Dirac equation and Klein-Gordon equation are used. These two equations combine relativity and quantum principles. In this research, we will replace flat spacetime in the Dirac equation and Klein-Gordon equation with Kerr spacetime. Kerr spacetime describes the effects of gravity on Fermino and Boson particles. To determine the effect of gravity, a neutron interferometer is used through the principle of phase shift. The Hamiltonian value will be obtained. In the Dirac equation, the effect of gravity only appears on the Hamiltonian H_1, H_5 and H_6. The phase shift values are $\Delta\beta_1 = 1.03091 \times 10^{-33}$, $\Delta\beta_5 \; -0.71764 \times 10^{-39}$ dan $\Delta\beta_6 = 7.2827496 \times 10^{-36}$. In the Klein-Gordon equation, the effect of gravity only appears on the Hamiltonian H_1. The phase shift value is $\Delta\beta_1 = 1.03091 \times 10^{-33}$. The Dirac equation contains more Hamiltonian terms that are not found in the Klein-Gordon equation. The more Hamiltonian terms, the more confounding Hamiltonian is in it. Confounding Hamiltonian will appear when the calculation involves the quantum part. From the calculation results, it is found that the Dirac equation has better accuracy than the Klein-Gordon equation when viewed from the calculation results of each phase shift.

Keywords: Dirac Equation; Klein-Gordon Equation; Kerr Geometry; Neutron Interferometer.

Introduction

The equations that can combine gravity and quantum are the Dirac equation and the Klein-Gordon equation in the form of the relativity wave equation. The Dirac equation was derived by Paul Dirac in 1928. The Dirac equation is derived from the Schrödinger equation, which is a partial differential equation of the wave function. The Dirac equation can explain the movement of particles with spin, such as electrons, protons, neutrinos, muons, quarks and their antiparticles. The Klein-Gordon equation was derived by Oskar Klein and Walter Gordon in 1926. This equation can explain the movement of spherical spin particles such as photons, gluons, phonons, W bosons and Z bosons.[1]

The quantum mechanics approach is mostly done with theoretical experiments, such as the search for solutions to the Dirac equation in Kerr-Newman spacetime.[2] However, as technology developed, it was possible to experiment in the laboratory and try to interpret quantum mechanics. It would be interesting to link gravity and quantum mechanics. So, in this case, the development of the neutron interferometer has an influential role. However, it does not yet have a fundamental amalgamation of quantum mechanics and gravity. So, it is necessary to observe the effects of gravity on quantum mechanics.

Research conducted by Overhauser and Colella (1974) proposed an experiment to examine the effect of the gravitational field from the earth on phase shift. This experiment was successfully carried out using a neutron interferometer. Based on the experiments that have been carried out, the phase difference in the gravitational potential produces the same value as the theoretical prediction with an error of 1% without distinguishing the inertial

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mass of the gravitational mass of the neutron. These results support the correctness of the equivalence principle at the quantum level. The atomic interferometer approach will increase the accuracy of a measurement. The Colella-Overhauser-Werner experiment using a quantum interferometer produces a sensitivity of 10^{-2} g, where g is the acceleration due to gravity.[3]

The gravitational effect applied to the quantum interferometer will produce a disturbance phenomenon. The phenomenon of disturbance is fundamentally different in classical and quantum physics. Quantum perturbation has a large impact on phase shift because the observed phase shift provides information on the external field that distorts the wave function of the particle. The importance of phase shifts by presenting electromagnetic potentials suggested by Ahanrov and Bohn on quantum perturbations was later proven experimentally.[4]

Research has shown that other effects are caused by the rotation of the earth and have been detected by Dresden and Yang (1979). This effect is similar to the Sagnac effect in a quantum interferometer. With this theoretical approach, other types of effects have been derived, such as optical analogies,[5] the eiconal relativity approach, the WKB approach,[6] the Doppler effect of media displacement,[7] and the analogous Aharnovon-Bohn effect. Anandan and Chiao proposed a gravitational radiation antenna using the Sagnac effect. The detection of gravity does not only state the proof of general relativity. But it also opened a new understanding of astronomical observations.

The phase shift effect has been derived using various approaches. Research conducted by Anandan (1977) calculated the phase shift caused by gravity and rotation in quantum disturbances using the Schwarzschild field,[8] Dresden and Yang (1979) also carried out the same thing but with a different approach. This study calculates the phase shift caused by the rotation of the neutron / optical interferometer derived from the Doppler effect caused by the displacement of the source and the displacement of the crystal reflection.[5] The reduction of the phase shift in quantum mechanics was also carried out by Sakurai (1980) by being influenced by the rotation of the earth.[9] Wajima et al. (1997) used the Schrödinger equation derived from spherical spin particles and half spin particles. This research resulted in the gravitational effect of the rotating earth on a quantum interferometer.[10]

The effect of gravity applied to a neutron interferometer can be carried out using two different types of equations. This research will study the effect of gravity on the neutron interferometer using the Dirac equation, which represents fermion particles and the Klein-Gordon equation for the boson particle. Then, we will look for an equation that represents the effect of gravity on fermion and boson particles. This equation will be applied to the neutron interferometer to determine the effect of gravity on the phase shift. This study uses Kerr Geometry, which will represent the effects of Earth's gravity and its rotation. These two disturbance effects use two different equations, so the differences will be analyzed.

Methods

The method used in this research is the analytical method by deriving the Dirac equation and Klein-Gordon equation in Kerr spacetime.

1. Changing the spacetime structure of the Dirac equation and the Klein-Gordon equation into a curved spacetime form
2. Determine the effect of gravity using the phase shift equation of the neutrino interferometer
3. Comparing the gravitational effect between the Dirac equation and the Klein-Gordon equation.

Result and Discussion

1. The Dirac equation in Kerr spacetime

1.1 Kerr Spacetime

The gravitational field that appears is caused by the object rotating. We assume that

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the Kerr metric represents the external field of this object. Variable limitation is done on slow rotation, weak field, metric form, and coordinate form (3+1)

$$ds^2 = \Big[c^2 + 2\phi - \omega_o^2(x^2+y^2) + \frac{2\phi^2}{c^2} + \frac{8GMR^2}{5c^2}\omega_o\omega_s(x^2+y^2) + \frac{2\phi}{c^2}\omega_o^2(x^2+y^2)\Big]dt^2 \quad (1.1)$$

$$-\left[\omega_o - 2\frac{\phi}{c^2}\omega_o - \frac{4GMR^2}{5c^2r^3}\omega_s\right](xdy - ydx)dt - \left(1-\frac{2\phi}{c^2}\right)(dx^2+dy^2+dz^2)$$

Using the metrics above, we can calculate the Christoffel symbol. The Christoffel symbol will be used to define the components of the connection spin.

Using the form (3+1) then the metric form $g_{\alpha\beta}$ in equation (1.1) is separated by following the following equation

$$g_{oo} = N^2 - \gamma_{ij}N^iN^j \quad (1.2)$$

$$g_{oi} = -\gamma_{ij}N^j \equiv -N_i \quad (1.3)$$

$$g_{ij} = -\gamma_{ij} \quad (1.4)$$

where N is the Lapse function, $N^i$is the vector shift, and γ_{ij} is the spatial metric on the 3D hypersurface. Define γ^{ij} the inverse matrix of γ_{ij}. Mertik $g^{\alpha\beta}$ can be separated according to the following equation

$$g^{oo} = \frac{1}{N^2} \quad (1.5)$$

$$g^{oi} = -\frac{N^i}{N^2} \quad (1.6)$$

$$g_{ij} = -\frac{N^iN^j}{N^2} - \gamma^{ij} \quad (1.7)$$

Equation (1.1) can be separated from the lapse parts, vector shifts, and spatial metrics to be[11]

$$N = c\left(1 + \frac{\phi}{c^2} + \frac{\phi^2}{2c^4}\right) \quad (1.8)$$

$$N^x = -\left(\omega_o - \frac{4GMR^2}{5c^2r^3}\omega_s\right)y \quad (1.9)$$

$$N^y = \left(\omega_o - \frac{4GMR^2}{5c^2r^3}\omega_s\right)x \quad (1.10)$$

$$N^z = 0 \quad (1.11)$$

$$\gamma_{ij} = \left(1 - \frac{2\phi}{c^2}\right)\delta_{ij} \quad (1.12)$$

to calculate the spin component of the connection, we need the following tetrads:

$$e^{\mu}_{(0)} = c\left(\frac{1}{N}, -\frac{N^i}{N}\right) \quad (1.13)$$

$$e^{\mu}_{(k)} = \left(0, e^{\mu}_{(k)}\right) \quad (1.14)$$

where the spatial three $e^i_{(k)}$ is defined as

$$\gamma_{ij} = e^i_{(k)}e^j_{(l)} = \delta_{kl} \quad (1.15)$$

by inserting equations (1.8) - (1.12) into equations (1.13) and (1.14), we get tetrads with a certain index

$$e^0_{(0)} = 1 - \frac{\phi}{c^2} \quad (1.16)$$

$$e^1_{(0)} = \left(\omega_o - \frac{\phi}{c^2}\omega_o - \frac{4GMR^2}{5c^2r^3}\omega_s\right)y \quad (1.17)$$

$$e^2_{(0)} = -\left(\omega_o - \frac{\phi}{c^2}\omega_o - \frac{4GMR^2}{5c^2r^3}\omega_s\right)x \quad (1.18)$$

$$e^3_{(0)} = 0 \quad (1.19)$$

$$e^i_{(j)} = \left(1 + \frac{\phi}{c^2}\right)\delta^i_j \quad (1.20)$$

From the above equation, the components of the spin connection can be calculated.[11]

1.2 Covariant Differentiation

By connecting the local inertial coordinate system with the general non-inertial coordinate system, we get the theoretical expansion of a field in curved spacetime. In accordance with the same strong principle, it states that all natural laws in an inertial coordinate system will be the same as in a Cartesian coordinate system that is not accelerated or does not have the influence of gravity. This approach is worth considering and using.

By using the tetrad form and forming a local inertial coordinate system ξ_x^a at each point in spacetime X. The metric in the non-inertial coordinate system takes the form. [11]

$$g_{\alpha\beta}(x) = e_\alpha^{(a)}(x) e_\beta^{(b)}(x)\, \eta_{ab} \tag{1.21}$$

where

$$e_\alpha^{(a)}(X) \equiv \left(\frac{\partial \xi_x^a(x)}{\partial x^\alpha}\right)_{x=X} \tag{1.22}$$

The equivalence principle in general relativity must be applied to all local inertial frames with scalar field components $\bar{A}^a, \bar{B}^a{}_b$, and so on. The scalar field definition can be chosen freely in the local inertial coordinate system but must contain a vector or tensor satisfying the Lorentz transformation $\Lambda_b^a(x)$ on x:

$$\bar{A}^a(x) \to \Lambda_b^a(x) \bar{A}^b(x) \tag{1.23}$$

$$\bar{B}^a{}_b(x) \to \Lambda_c^a(x) \Lambda_b^d(x) \bar{B}^c{}_d(x) \tag{1.24}$$

In general the field $\bar{\psi}_m(x)$ is defined in the local inertial coordinate system which changes in the following way

$$\bar{\psi}_m(x) \to \sum_n [U(\Lambda(x))]_{mn} \bar{\psi}_n(x) \tag{1.25}$$

Where $U(\Lambda(x)$ is the matrix of the Lorentz grub, for example if $\bar{\psi}$ is the covariance of vector $\bar{A}_a$, so $U\big(\Lambda(x)\big)$ is equal to

$$[U(\Lambda(x))]_a^b = \Lambda_a^b(x) \tag{1.26}$$

whereas for the contravariant tensor $\bar{T}^{ab}$

$$[U(\Lambda(x))]_{cd}^{ab} = \Lambda_c^a(x) \Lambda_d^b(x) \tag{1.27}$$

The vector coordinates are transformed into the generalized coordinate transformation $\to x'$:

$$\frac{\partial}{\partial x^\alpha} \to \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta} \tag{1.28}$$

Transformed scalar coordinate derivative with general field using Lorentz transform rule (4.47)

$$\begin{aligned} & e_{(a)}^\alpha(x) \frac{\partial}{\partial x^\alpha} \bar{\psi}(x) \\ & \to \Lambda_a^b(x) e_{(b)}^\alpha \frac{\partial}{\partial x^\alpha} \{U(\Lambda(x)) \bar{\psi}(x)\} \\ & = \Lambda_a^b(x) e_{(b)}^\alpha(x) \left[U\big(\Lambda(x)\big) \frac{\partial}{\partial x^\alpha} \bar{\psi}(x) \right. \\ & \left. + \left\{ \frac{\partial}{\partial x^\alpha} U(\Lambda(x)) \right\} \bar{\psi}(x) \right] \end{aligned} \tag{1.29}$$

We define an operator $\bar{D}_a$ that satisfies a position-dependent Lorentz transformation. This operator satisfies the transformation law

$$\bar{D}_a \bar{\psi}(x) \to \Lambda_a^b(x) U\big(\Lambda(x)\big) \bar{D}_b \bar{\psi}(x) \tag{1.30}$$

By replacing $\partial_a \bar{\psi}(x)$ in the field equation in flat spacetime to $\bar{D}_a \bar{\psi}(x)$ we will get a field equation that does not depend on the inertial coordinate system. [10]

By considering equation (1.29), it will be obtained the scalar coordinates of the derivative of the Lorentz vector $\bar{D}_a$ with the form:

$$\bar{D}_a e_{(a)}^\alpha \equiv \left[\frac{\partial}{\partial x^\alpha} - \Gamma_\alpha \right] \tag{1.31}$$

Where Γ_α is a matrix that satisfies Lorentz transformation law

$$\begin{aligned} & \Gamma_\alpha(x) \\ & \to U\big(\Lambda(x)\big) \Gamma_\alpha(x) U^{-1}\big(\Lambda(x)\big) \\ & + \left[\frac{\partial}{\partial x^\alpha} U\big(\Lambda(x)\big) \right] U^{-1}(\Lambda(x)) \end{aligned} \tag{1.32}$$

The results of this derivation discuss a general field containing a spinor inserted into curved spacetime.

1.3 Connection

Using the Lorentz transform law a field denoted by ψ_m is transformed using the Lorentz transform by Λ_b^a

$$\psi'_m = \sum_n [U(\Lambda)]_{mn}\,\psi_n \tag{1.33}$$

Considering a small Lorentz group, we use a Lorentz transform that satisfies the following identity equation :

$$\Lambda_b^a(x) = \delta_b^a + \omega_b^a(x), \quad |\omega_b^a| \ll 1 \tag{1.34}$$

The matrix describing $U(\Lambda)$ must be very close to the following Identity :

$$U\big(1+\omega(x)\big) = \mathbf{1} + \frac{1}{2}\omega^{ab}(x)\sigma_{ab}) \tag{1.35}$$

If we consider the covariance vector $\bar{A}_a$, we get

$$[\sigma_{ab}]_c^d = \eta_{ac}\delta_b^d - \eta_{bc}\delta_a^d \tag{1.36}$$

The matrix σ_{ab} must be limited by the law of multiplication of grub $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$. If it is applied to the product $\Lambda(1+\omega)\Lambda^{-1}$ then :

$$U(\Lambda)U(1+\omega)U(\Lambda^{-1}) = U(1+\Lambda\omega\Lambda^{-1}) \tag{1.37}$$

with minimize ω

$$U(\Lambda)\,\sigma_{ab}\,U(\Lambda^{-1}) = \sigma_{cd}\Lambda_a^c\Lambda_b^d \tag{1.38}$$

If grouped according to $\Lambda = 1+\omega$ and $\Lambda^{-1} = 1-\omega$, we get

$$\omega^{cd}\big[\sigma_{ab,}\sigma_{cd}\big] = \omega^{cd}\,(\eta_{cb}\sigma_{ad} - \eta_{ca}\sigma_{bd} + \eta_{db}\sigma_{ca} - \eta_{da}\sigma_{cb}) \tag{1.39}$$

Then we get the commutation form

$$[\sigma_{ab},\sigma_{cd}] = \eta_{cb}\sigma_{ad} - \eta_{ca}\sigma_{bd} + \eta_{db}\sigma_{ca} - \eta_{da}\sigma_{cb} \tag{1.40}$$

and the contravariant tensor $\bar{T}^{ab}$

$$[\sigma_{ab}]_{ef}^{cd} = \delta_a^c\eta_{be}\delta_f^d - \delta_b^c\eta_{ae}\delta_f^d + \delta_a^d\eta_{bf}\delta_e^c - \delta_b^d\eta_{af}\delta_e^c \tag{1.41}$$

Using the transformation in equation (1.32) by fulfilling the Lorentz Infinitesimal transformation (1.34), the connecting form Γ_α transforms to[10]

$$\Gamma_\alpha(x) \to \Gamma_\alpha(x) + \frac{1}{2}\omega^{ab}(x)\,[\sigma_{ab},\Gamma_\alpha(x)] + \frac{1}{2}\sigma_{ab}\,\frac{\partial}{\partial x^\alpha}\omega^{ab}(x) \tag{1.42}$$

defined form of connection

$$\Gamma_\alpha(x) \to \frac{1}{2}C_\alpha^{ab}(x)\sigma_{ab} \tag{1.43}$$

$C_\alpha^{ab}(x)$ is antisymmetric in a and b. By using the transformation law (1.42) and the commutation relationship (1.41), we get $C_\alpha^{ab}(x)$

$$C_\alpha^{ab}(x) \to C_\alpha^{ab}(x) + \omega_c^a(x)C_\alpha^{cb}(x) + \omega_c^b(x)\,C_\alpha^{ac} + \frac{\partial}{\partial x^\alpha}\omega^{ab}(x) \tag{1.44}$$

The derivative relationship of $\bar{D}_a$ and tetrad $e_{(a)\alpha}$.

$$\bar{D}_a e_{(b)\alpha} \equiv e_{(a)}^\mu\left[\frac{\partial}{\partial x^\mu}e_{(b)\alpha} - \eta_{bc}C_\mu^{cd}\,e_{(d)\alpha}\right] \tag{1.45}$$

Then the total derivative of covariance $\bar{D}_a$ is defined as[10]

$$\bar{D}_a e_{(b)\alpha} \equiv e_{(a)}^\mu\left[\frac{\partial}{\partial x^\mu}e_{(b)\alpha} - \eta_{bc}C_\mu^{cd}e_{(d)\alpha} - \Gamma_{\alpha\mu}^v\,e_{(b)v}\right] \tag{1.46}$$

If derived in the same way in the tetrad $e_\alpha^{(a)}$ then we get

$$\bar{D}_a e_\alpha^{(b)} \equiv e_{(a)}^\mu\left[\frac{\partial}{\partial x^\mu}e_\alpha^{(b)} - \eta_{bc}C_\mu^{cd}e_\alpha^{(b)} - \Gamma_{\alpha\mu}^v\,e_\alpha^{(b)}\right] \tag{1.47}$$

the covariance derivative of the metrics $g_{\alpha\beta}$ is zero

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$$\nabla_\mu g_{\alpha\beta} \equiv 0 \tag{1.48}$$

therefore from equation (1.21) becomes

$$e_\mu^{(a)}\left[\left(\bar{D}_\alpha e_\alpha^{(b)}\right)e_{(b)\beta} + e_\alpha^{(b)}\left(\bar{D}_\alpha e_{(b)\beta}\right)\right] = 0 \tag{1.49}$$

The simple solution of the above equation is

$$\bar{D}_\alpha e_\alpha^{(b)} = \bar{D}_\alpha e_{(b)\alpha} = 0 \tag{1.50}$$

Using this condition, we will look for the relationship Γ_α and tetrad $\mathrm{e}_{(a)\alpha}$ from equations (1.46) and (1.47)

$$C_\alpha^{ab}(x) = -\eta^{ac}\eta^{bd} e_{(c)}^{\lambda} \nabla_\alpha e_{(d)\lambda} \tag{1.51}$$

therefore, the connector Γ_α

$$\Gamma_\alpha(x) = -\frac{1}{2}\sigma^{ab} g_{\mu v} e_{(a)}^{\mu} \nabla_\alpha e_{(b)}^{v} \tag{1.52}$$

according to the small Lorentz transform (4.59) the tetrad $e_{(a)\alpha}$ changes accordingly

$$\mathrm{e}_{(a)\alpha}(x) \to \mathrm{e}_{(a)\alpha}(x) + \omega_a^b(x)\mathrm{e}_{(b)\alpha}(x) \tag{1.53}$$

so

$$\begin{aligned} e_{(a)}^{\lambda}(x)\nabla_{(\alpha)}\mathrm{e}_{(b)\lambda}(x) \to {} & e_{(a)}^{\lambda}(x)\nabla_{(\alpha)}\mathrm{e}_{(b)\lambda}(x) \\ & +\omega_a^c(x) e_{(c)}^{\lambda}\nabla_{(\alpha)}\mathrm{e}_{(b)\lambda}(x) \\ & + \omega_b^c(x) e_{(a)}^{\lambda}\nabla_{(\alpha)}\mathrm{e}_{(c)\lambda}(x) \\ & - \frac{\partial}{\partial x^\alpha}\omega_{ab}(x) \end{aligned} \tag{1.54}$$

1.4 Dirac equation in Kerr spacetime

The covariance derivative $\bar{D}_a$ is used to derive the Dirac equation in curved spacetime. This equation is formed from the inertial coordinates ξ_x^a

$$\left[i\hbar\,\gamma^{(a)}\,\frac{\partial}{\partial \xi_x^a} - mc\right]\bar{\psi}(\xi) = 0 \tag{1.55}$$

Replacing the derivative $\partial/\partial\xi_x^a$ with the covariant derivative $\bar{D}_a$ in equation (1.55) to replace flat spacetime into curved spacetime

$$\left[i\hbar\,\gamma^{(a)} e_{(a)}^{\alpha}\left(\frac{\partial}{\partial x^\alpha} - \Gamma_\alpha\right) - mc\right]\bar{\psi}(\xi) = 0 \tag{1.56}$$

then the Dirac equation can be derived in the form of the general covariance.[10]

$$\left[i\hbar\gamma^{(a)}D_a - mc\right]\psi(x) = 0 \tag{1.57}$$

Transforming the spinor using the Lorentz transformation Λ then we get

$$\psi'(x) = S(\Lambda)\psi(x) \tag{1.58}$$

Where $S(\Lambda)$ is the matrix of the Lorentz grub. Then equation (1.57) becomes

$$\begin{aligned} \big[i\hbar S(\Lambda)\gamma^{(a)} S^{-1}(\Lambda)(\Lambda^{-1})_a^b\,\partial_{b'} & \\ - mc\big]\,\psi'(x) = 0 & \end{aligned} \tag{1.59}$$

In order for the Dirac equation to be a covariant form that satisfies the Lorentz transformation, it must follow the following relationship

$$i\hbar S(\Lambda)\gamma^{(a)}S^{-1}(\Lambda) = (\Lambda^{-1})_b^a\gamma^{(b)} \tag{1.60}$$

using Lorentz infinitesimal transformation (1.34) then $S(\Lambda)$ must be written in the following form

$$S(\Lambda) = 1 + \frac{1}{2}\omega^{ab}\sigma_{ab} \tag{1.61}$$

from equation (1.60) can be derived by following the following conditions

$$\frac{1}{2}\omega^{ab}\left[\sigma^{ab},\gamma^{(c)}\right] = -\omega_d^c\gamma^{(d)} \tag{1.62}$$

In the case of a spinor, this condition is set by

$$\sigma^{ab} = \frac{1}{4}\left[\gamma^{(a)},\gamma^{(b)}\right] \tag{1.63}$$

from equation (1.52) we get spin connection.[10]

$$\Gamma_\alpha(x) = -\frac{1}{8}\left[\gamma^{(a)},\gamma^{(b)}\right]g_{\mu v}e_{(a)}^{\mu}\nabla_\alpha e_{(b)}^{v} \tag{1.64}$$

1.4 Equation of Schrödinger-type

Derive the Schrödinger type equation from the Dirac covariance equation (1.57) using a certain tetrad, then the Dirac covariance metric γ^α can be written as

$$\gamma^0 = \gamma^{(a)} e^0_{(a)} = \gamma^{(0)} \frac{c}{N} \tag{1.65}$$

$$\gamma^0 = \gamma^{(a)} e^i_{(a)} = -\gamma^{(0)} \frac{c}{N} N^i + \gamma^{(j)} e^i_{(j)} \tag{1.66}$$

Replacing D_a with $\left(\frac{\partial}{\partial x^\alpha} - \Gamma_\alpha\right)$ then the Dirac covariance equation in equation (1.57) becomes

$$\left[i\hbar\gamma^{(a)} \frac{\partial}{\partial x^\alpha} - i\hbar\gamma^{(a)}\Gamma_\alpha - mc\right] \psi(x) = 0 \tag{1.67}$$

The equation above is a Dirac equation whose spacetime has been converted into Kerr spacetime. This equation describes a fermion particle that moves in a curved spacetime and undergoes a rotation. Equation (1.67) needs to be separated from the space and time terms in order to obtain an equation of the Schrödinger type by entering equation (1.65) and equation (1.66)

$$i\hbar\gamma^{(0)} \frac{c}{N} \frac{\partial}{\partial t} \psi = \Big[\Big(-\gamma^{(0)} \frac{c}{N} N^i + \gamma^{(j)} e^i_{(j)}\Big)\Big(-i\hbar \frac{\partial}{\partial x^i} + i\hbar\Gamma_i\Big) + i\hbar\gamma^{(0)} \frac{\mathrm{c}}{\mathrm{N}} \Gamma_0 + mc\Big]\psi \tag{1.68}$$

Multiplying by $\gamma^{(0)} cN$, we get the Schrödinger type equation which has separate terms between the space and time parts.

$$i\hbar \frac{\partial}{\partial t} \psi = [(\gamma^{(0)}\gamma^{(j)} cNe^i_{(j)} - N^i)(\bar{\boldsymbol{p}}_\iota + i\hbar\Gamma_i) + i\hbar\Gamma_0 + \gamma^{(0)} mc^2 N]\psi \tag{1.69}$$

The Dirac equation in Kerr spacetime describes a fermion particle that moves in a curved and rotating spacetime. While the usual Dirac equation only moves in flat spacetime.

1.6 Non-relativistic Hamitonian

The spinor and the Hamiltonian are redefined as follows

$$\psi' = \gamma^{\frac{1}{4}} \psi, \qquad H' = \gamma^{\frac{1}{4}} H \gamma^{-\frac{1}{4}} \tag{1.70}$$

where γ is the determinant of the spatial matrix

$$\gamma = \det(\gamma_{ij}) \tag{1.71}$$

because it produces a scalar invariant then

$$(\psi, \varphi) \equiv \int \bar{\psi}\ \varphi \sqrt{\gamma} d^3x \tag{1.72}$$

The definition of a scalar product is the same as in flat spacetime with the following redefinition

$$\langle \psi', \varphi' \rangle \equiv \int \sqrt{\bar{\psi'}}\ \varphi' d^3x \tag{1.73}$$

Then, the non-relativity Hamiltonian will be obtained using the FWT transformation on the Hamiltonian H'.

$$\bar{H}' = \begin{pmatrix} \bar{H}'_+ & 0 \\ 0 & \bar{H}_- \end{pmatrix} + O\left(\frac{1}{c^4}\right) \tag{1.74}$$

Then the Hamiltonian is defined by subtracting

$$H_+ \equiv \bar{H}_+ - mc^2 \tag{1.75}$$

Using the above definition, we get the Schrödinger equation with general relativity correction for the large component

$$i\hbar \frac{\partial}{\partial t} \Phi = \Big[\frac{\bar{\boldsymbol{p}}^2}{2m} + m\phi - \boldsymbol{\omega}.(\boldsymbol{L} + \boldsymbol{S}) + \frac{1}{c^2}\Big(\frac{4GMR^2}{5r^3} \omega.(\boldsymbol{L} + \boldsymbol{S}) - \frac{\bar{\boldsymbol{p}}^4}{8m^3} + \frac{1}{2}\phi^2 + \frac{3}{2m}\bar{\boldsymbol{p}}.\phi\bar{\boldsymbol{p}}\Big) + \frac{1}{c^2}\Big(\frac{3GM}{2mr^3}\boldsymbol{L}.\boldsymbol{S} + \frac{6GMR^2}{5r^5}\boldsymbol{S}.[\boldsymbol{r}\ x\ (\boldsymbol{r}x\boldsymbol{\omega})]\Big)\Big]\Phi \tag{1.76}$$

Following the canochial quantity procedure, a non-relativity Hamiltonian will be obtained with $\boldsymbol{S} = \boldsymbol{0}$

$$i\hbar\frac{\partial}{\partial t}\phi' = \left[\frac{\overline{\boldsymbol{p}}^{2}}{2m} + m\phi - \boldsymbol{\omega}\cdot\boldsymbol{L} + \frac{1}{c}\left(\frac{4GMR^2}{5r^3}\omega\cdot\boldsymbol{L} - \frac{\overline{\boldsymbol{p}}^{4}}{8m^3} + \frac{1}{2}m\phi^2 + \frac{3}{2m}\overline{\boldsymbol{p}}.\phi\overline{\boldsymbol{p}}\right)\right] \tag{1.77}$$

1.5 Gravitational Effects in Neutron Interferometer

The neutron interferometer describes the movement of fermion particles through a curved and rotating spacetime. Curved spacetime is caused by gravity. The total Hamiltonian of the Dirac equation in Kerr spacetime consists of several terms

$$i\hbar\frac{\partial}{\partial t}\phi' = \left[\frac{\overline{\boldsymbol{p}}^{2}}{2m} + m\phi - \boldsymbol{\omega}\cdot\boldsymbol{L} + \frac{1}{c}\left(\frac{4GMR^2}{5r^3}\omega\cdot\boldsymbol{L} - \frac{\overline{\boldsymbol{p}}^{4}}{8m^3} + \frac{1}{2}m\phi^2 + \frac{3}{2m}\overline{\boldsymbol{p}}.\phi\overline{\boldsymbol{p}}\right)\right] \tag{1.78}$$

If we use the wave function Φ_0 in the Schrödinger equation, we can solve it by the following equation

$$i\hbar\frac{\partial}{\partial t}\Phi_0 = H_0\,\Phi_0 \tag{1.79}$$

The total Hamiltonian in the Schrödinger type equation (1.77) can be solved using the wave function

$$\Phi = \Phi_0 \exp\left(i\sum_k \beta_k\right) \tag{1.80}$$

where β_k contains the Hamiltonian of the disturbance term

$$\beta_k = \frac{1}{\hbar}\int^{t} \Delta H_k dt \tag{1.81}$$

Two neutron waves that pass through the ABD path and the ACD path meet at point D. the meeting point at D the phase shift can be calculated as follows

$$\Delta\beta_k = \beta_{k(ACD)} - \beta_{k(ABD)} = -\frac{1}{\hbar}\oint \Delta H_k dt \tag{1.82}$$

each Hamiltonian term of equation (1.77) will be calculated the phase shift using equation (1.81). Then will be evaluated the part that causes the phase shift.[10]

$$\Delta\beta_0 = 0 \tag{1.83}$$

$$\Delta\beta_1 \cong \frac{m^2 gA\lambda}{2\pi\hbar^2}\sin\varphi \tag{1.84}$$

$$\Delta\beta_2 = \frac{2m}{\hbar}\boldsymbol{\Omega}\cdot\boldsymbol{A} \tag{1.85}$$

$$\Delta\beta_3 = \frac{1}{5}\Delta\beta_2\frac{r_g}{5R}\left[1 - \frac{3}{\boldsymbol{\omega}}\left(\frac{\boldsymbol{R}}{R}\omega\right)\frac{\boldsymbol{R}}{R}\boldsymbol{A}\right] \tag{1.86}$$

$$\Delta\beta_4 = 0 \tag{1.87}$$

$$\Delta\beta_5 = -\frac{1r_g}{2R}\Delta\beta_1 \tag{1.88}$$

$$\Delta\beta_6 = \frac{3}{2}\left(\frac{\lambda_c}{\lambda}\right)^2\Delta\beta_1 \tag{1.89}$$

2. Klein Gordon Equation in Kerr Spacetime

2.1 Klein Gordon equation in Kerr spacetime

The Klein-Gordon equation describes the Boson particle, which has zero spin. If the spacetime of the Klein-Gordon equation is replaced by the Kerr spacetime, it will affect the Boson particle. To find out the effect, we will first derive the form of the Klein-Gordon equation in Kerr spacetime, and then the equation will be applied to the Neutron interferometer. Starting with the Lorenz covariance equation from the Klein-Gordon equation

$$\nabla^{\mu}\nabla_{\mu}\phi-\left(\frac{mc}{\hbar}\right)^{2}\phi=0 \qquad (2.1)$$

With $mc^2 = 0$ in the case of rest energy then

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}+2i\frac{m}{\hbar}\frac{\partial\psi}{\partial t}+\nabla^2\psi=0 \qquad (2.2)$$

where ∇^2 the Lapasian operator, three-dimensional. The relative order of the first term in the other (4.147) equation is $O((v/c)^2)$, where v represents the speed characteristic of the wave packet or is equivalent to $O((\lambda_0/\lambda)^2)$. Where λ_0 is the Compton wavelength and λ is the de Broglie wavelength of the particle.[11]

Equation (2.2) can be reduced to the ordinary Schrödinger equation.[12]

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi \qquad (2.3)$$

Assuming the external gravitational field of the earth with the metric Kerr. At Boyer-Lindquist coordinates given the equation of the line

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\rho^2}(cdt-a\sin^2\theta\, d\phi)^2 \\ &+\frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi-acdt]^2 \qquad (2.4) \\ &+\rho^2\left(\frac{dr^2}{\Delta}+d\theta^2\right) \end{aligned}$$

The form of the d'Alembertian operator in Boyer-Linduquist coordinates can be calculated by the form

$$\square_g=\frac{1}{|g|^{1/2}}\frac{\partial}{\partial x^{\boldsymbol{a}}}\left(|g|^{1/2}g^{\boldsymbol{ab}}\frac{\partial}{\partial x^{\boldsymbol{b}}}\right) \qquad (2.5)$$

where $|g|$ is $|det\ g\,|$ then get

$$\begin{aligned} \square_g = &\frac{\sigma^2}{\Delta\rho^2}\frac{\partial^2}{\partial t^2}+\frac{4aMr}{\Delta\rho^2}\frac{\partial^2}{\partial t\partial\varphi} \\ &-\frac{1}{\rho^2}\frac{\partial}{\partial r}\left(\Delta\frac{\partial}{\partial r}\right)-\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} \qquad (2.6) \\ &-\frac{\rho^2-2Mr}{\rho^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \end{aligned}$$

Then the form of the three-dimensional Laplacian operator is

$$\begin{aligned} \nabla^2 = &-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \\ &-\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \qquad (2.7) \\ &-\frac{2Mr}{(r^2\sin^2\theta+a^2\cos^2\theta\sin^2\theta)}\frac{\partial^2}{\partial\varphi^2} \end{aligned}$$

Entering equation (2.7) into equation (2.3) then the Klein Gordon equation becomes

$$\begin{aligned} &\frac{i\hbar\partial\psi}{\partial t} \\ &=-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right. \\ &-\frac{1}{r^2\hbar^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right. \qquad (2.8) \\ &\left.\left.+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]\psi \\ &-\frac{GMm}{r}\psi+\frac{2GMa}{r^3c}\left(-i\hbar\frac{\partial}{\partial\phi}\right)\psi \end{aligned}$$

Transformation of coordinates $\phi \rightarrow \phi - \omega t$, where ω is the angular velocity of the earth, the equation (2.8) becomes

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t}= &-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{L^2}{r^2\hbar^2}\right]\psi \\ &-\frac{GMm}{r}\psi-\omega L_z\psi+\frac{2GMa}{r^3c}L_z\psi \end{aligned} \qquad (2.9)$$

2.1 Gravitational Effects in Neutron Interferometer

The effect of gravity on the Klein-Gordon equation can be seen by applying equation (2.9) to the neutron interferometer. Equation (2.9) consists of two types of Hamiltonian, namely the main Hamiltonian and the confounding Hamiltonian

$$H_0=-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{L^2}{2mr^2} \qquad (2.10)$$

$$H_1 = -\frac{GMm}{r} \equiv V_g \quad (2.11)$$

$$H_2 = -\omega L_z \quad (2.12)$$

$$H_3 = \frac{2GMa}{r^3 c} L_2 \quad (2.13)$$

The wave entering the neutron interferometer is defined as follows

$$\beta_k = \frac{1}{\hbar} \int^t \Delta H_k dt \quad (2.14)$$

At the meeting point at D, the phase shift can be calculated as follows:

$$\Delta\beta_k = \beta_{k(ACD)} - \beta_{k(ABD)} = -\frac{1}{\hbar} \oint \Delta H_k dt \quad (2.14)$$

Each Hamiltonian is calculated using equation (2.14), so the value of is obtained

$$\Delta\beta_0 = 0 \quad (2.15)$$

$$\Delta\beta_1 = \frac{m^2 g A \lambda}{\hbar^2} \sin\varphi \quad (2.16)$$

$$\Delta\beta_2 = \frac{2m}{\hbar} \boldsymbol{\Omega} \cdot \boldsymbol{A} \quad (2.17)$$

$$\Delta\beta_3 = \frac{1}{5} \Delta\beta_2 \frac{r_g}{5R} \left[\boldsymbol{A} - \frac{3}{\boldsymbol{\omega}} \left(\frac{\boldsymbol{R}}{R} \right) \frac{\boldsymbol{R}}{R} \boldsymbol{A} \right] \quad (2.18)$$

Based on the results of the research, the form of the Dirac equation in Kerr spacetime takes the following form

$$\left[i\hbar\gamma^{(a)} \frac{\partial}{\partial x^\alpha} - i\hbar\gamma^{(a)} \Gamma_\alpha - mc \right] \psi(x) = 0 \quad (2.19)$$

Equation (3.1) is Dirac's equation with spacetime that has been converted into Kerr spacetime. This equation describes fermion particles that move in curved spacetime and experience rotation. When compared with the Dirac equation in flat spacetime, there is a difference in the Connection term Γ_α which causes curved spacetime Geometry. The Γ_α connection has a connection spin component resulting from the line equation ds^2 in Geometry Kerr. If we do an approximation with $\Gamma_\alpha = \Gamma_0$, we get the Dirac equation in flat spacetime again.

Entering the values in equation (1.83)-(1.89) from the experimental results, the value of the phase shift will be obtained as follows

$$\Delta\beta_0 = 0 \quad (2.20)$$

$$\Delta\beta_1 = 1.03091 \times 10^{-33} \quad (2.21$$

$$\Delta\beta_2 = 5.631229 \quad (2.22)$$

$$\Delta\beta_3 = -1.0726 \times 10^{-9} \quad (2.23)$$

$$\Delta\beta_4 = 0 \quad (2.24)$$

$$\Delta\beta_5 = -0.71764 \times 10^{-39} \quad (2.25)$$

$$\Delta\beta_6 = 7.2827496 \times 10^{-36} \quad (2.26)$$

The value of the phase shift above can be divided into two types based on the cause, namely the phase shift caused by the effect of gravity and the phase shift caused by rotation. The emergence of these two effects is due to the curved and rotating spacetime. In this study only focus on the effect of gravity only. So only on the Hamiltonian $H_1, H_5,$ and H_6 which appears the influence of gravity, the rest is influenced by rotation.

Although the value of the phase shift is very small, this value is enough to show that gravity can affect the fermion particles. Then, the results of the above phase shift calculations will be compared with the phase shift values from the Klein-Gordon equation. This comparison is the basis for determining the equation that experiences the greatest gravitational influence.

The form of the Klein-Gordon equation in Kerr spacetime takes the following form

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{L^2}{r^2 \hbar^2} \right] \psi - \frac{GMm}{r} \psi - \omega L_z \psi + \frac{2GMa}{r^3 c} L_z \psi \quad (2.27)$$

Equation (2.27) is the Klein-Gordon equation in Kerr spacetime. Using a non-relativistic approach, the form of the above equation is almost the same as the Schrödinger equation, with space and time terms separated. The difference between the Klein-Gordon equation in flat spacetime and the Klein-Gordon equation in Kerr spacetime lies in the metric. This metric represents the shape of spacetime.

Entering the values in equation (2.15)-(2.18) from the experimental results, the phase shift values will be obtained as follows:

$$\Delta\beta_0 = 0 \quad (2.28)$$

$$\Delta\beta_1 = 1.031 \times 10^{-33} \quad (2.29)$$

$$\Delta\beta_2 = 5.631229 \quad (2.30)$$

$$\Delta\beta_3 = -1.0726 \times 10^{-9} \quad (2.31)$$

The above phase shift values can be divided into two types based on the cause, namely the phase shift caused by the effects of gravity and the phase shift caused by rotation. The appearance of these two effects is due to the curved and rotating spacetime. In this study, it only focuses on the influence of the gravitational effect. So only the Hamiltonian H_1 shows the influence of gravity. The rest is influenced by rotation.

The phase shift value obtained is the same as the phase shift in the Dirac equation but differs only in the amount. The phase shift calculations show that the Dirac equation experiences a greater gravitational influence than the Klein-Gordon equation. The Dirac equation contains more Hamiltonian terms which are not found in the Klein-Gordon equation. The more Hamiltonian terms indicate that there are more Hamiltonian confounders in it. The disturbing Hamiltonian will appear when the calculation involves quantum terms.

Pergeseran fase $\Delta\beta_0$ merupakan pergeseran fase yang dihasilkan dari Hamiltonian klasik (non-kuantum) dilihat dari suku Hamiltoniannya yang memuat hamiltoninan klasik, sedangkan sisanya merupkan Hamiltonian kuantum dan Hamiltoninan relativistik. Perbedaan antara ruang waktu datar dan melengkung bisa dilihat dari nilai masing-masing pergeseran fasenya. Jika pergesern fase bernilai nol maka partikel berada pada ruang waktu datar sedangkan jika memiliki nilai maka partikel berada di ruang waktu melengkung.

The phase shift $\Delta\beta_0$ is a phase shift resulting from the classical Hamiltonian (non-quantum) seen from its Hamiltonian term, which contains classical Hamiltonian. At the same time, the rest are quantum Hamiltonian and relativistic Hamiltonian. The difference between flat and curved spacetime can be seen from the value of each phase shift. If the phase shift is zero, the particle is in a flat spacetime, whereas if it has a value, it is in a curved spacetime.

Conclusion

The effect of gravity causes a phase shift in the Dirac equation and Klein Gordon equation which is observed through the neutron interferometer. In the Dirac equation the effect of gravity is on the Hamiltonian terms $H_1, H_5,$ danH_6. The phase shift values are $\Delta\beta_1 = 1.03091 \times 10^{-33}$, $\Delta\beta_5 \; -0.71764 \times 10^{-39}$ dan $\Delta\beta_6 = 7.2827496 \times 10^{-36}$.

In the Klein-Gordon equation, the effect of gravity is on the Hamiltonian term H_1 only. The phase shift value is $\Delta\beta_1 = 1.03091 \times 10^{-33}$. The results of the phase shift calculation show that the Dirac equation has a greater gravitational effect than the Klein-Gordon equation.

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