Spectrum of the Laplacian Matrix of Non-commuting Graph of Dihedral Group D_{2n}

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ABSTRACT

Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_p\}$, $A(G)$ is adjacency matrix of G and $D(G)$ is diagonal matrix with entry $d_{ii} = \deg_G(v_i)$, $i = 1, 2, \ldots, p$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Spectrum of the Laplacian matrix is obtained by finding of eigenvalues of $L(G)$ and their multiplicities. In this paper we study spectrum of the Laplacian matrix of non-commuting graph of dihedral group $\Gamma_{D_{2n}}$, and give results about characteristics polyniomial of $L(\Gamma_{D_{2n}})$ and its spectrum of the Laplacian matrix. We obtained spectrum of the Laplacian matrix of $\Gamma_{D_{2n}}$ is

$$Spec_L(\Gamma_{D_{2n}}) = \begin{bmatrix} 2n-1 & n & 0 \\ n & n-2 & 1 \end{bmatrix}$$

Key-words: eigenvalues, eigenvector, spectrum, Laplacian matrix, non-commuting graph, dihedral group.

INTRODUCTION

A graph G is a finite nonempty set of objects called vertices together with a (possibly empty) set of unordered pairs of distinct vertices of G called edges (Cartrand & Lesniak, 1986). The vertex set of G is denoted by $V(G)$, while the edge set is denoted by $E(G)$.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. The adjacency matrix of G, denoted by $A(G)$, is $(p \times p)$-square matrix where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ if $v_iv_j \notin E(G)$, $1 \le i, j \le p$. The diagonal matrix of G, denoted by $D(G)$, is diagonal matrix where $d_{ii} = \deg_G(v_i)$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Since the Laplacian matrix is real and symmetric, all its eigenvalues μ_i, $i = 1, 2, 3, \ldots, p$ are nonnegative real numbers and can be labeled so that $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_p = 0$. If $\mu_{i_1} \ge \mu_{i_2} \ge \cdots \ge \mu_{i_g}$ are the distinct eigenvalues, then spectrum of $L(G)$ can be written as

$$Spec_L(G) = \begin{bmatrix} \mu_{i_1} & \mu_{i_2} & \ldots & \mu_{i_g} \\ m_1 & m_2 & \ldots & m_g \end{bmatrix}$$

where m_j indicates the algebraic multiplicity of the eigenvalue μ_{i_j} (Yin, 2008). Of course $m_1 + m_2 + \ldots + m_g = p$ (Ayyaswamy & Balachandran, 2010). The multiplicity of μ_{i_j} as a root of characteristic equation $det(\mu_{i_j}I - L(G)) = 0$ is equal to the dimension of the space of eigenvectors corresponding to μ_{i_j} (Bigg, 1974).

Let G be a non-abelian group with center $Z(G)$. Associate a graph Γ_G of G whose vertices are the non-central elements $G\backslash Z(G)$ and whose edges join those vertices $x, y \in G\backslash Z(G)$ for which $xy \neq yx$ in G. Then Γ_G is said to be the non-commuting graph of G (Abdollahi, et.al, 2006 and Abdollahi, et.al, 2010). Note that if G is abelian, then Γ_G is the null graph. Because G is non-abelian, the non-commuting graph Γ_G of G is always connected with diameter 2 and girth 3.

Some research about spectrum of the Laplacian matrix has been conducted. Yin (2008) investigated spectrum of the Laplacian matrix of graph G_l, where G_l obtained from complete graph K_l by adhering the root of isomorphic trees T to every vertex of K_l and d_{k-j+1} be the degree of viertices in the level j. Abdussakir, et.al (2012) determined spectrum of the Laplacian matrix of complete multipartite graph $K(\alpha_1, \alpha_1, \ldots, \alpha_n)$. In this paper we determined spectrum of the Laplacian matrix of non-commuting graph of dihedral group order $2n$, where n is odd natural numbers and $n \ge 3$, because there are no research in this topic until the day.

RESULTS

In this section, we will give an axample to determine spectrum of the Laplacian matrix of dihedral group for the cases of dihedral group

Dihedral group $D_6 = \{1, r, r^2, s, sr, sr^2\}$ with composition operation is non-abelian group. Using Cayley table

∘	**1**	***r***	***r²***	***s***	***sr***	***sr²***
1	1	*r*	*r²*	*s*	*sr*	*sr²*
r	*r*	*r²*	1	*sr²*	*S*	*sr*
r²	*r²*	1	*r*	*sr*	*sr²*	*s*
s	*s*	*sr*	*sr²*	1	*R*	*r²*
sr	*sr*	*sr²*	*s*	*r²*	1	*R*
sr²	*sr²*	*s*	*sr*	*r*	*r²*	1

we can have center of D_6 is $Z(D_6) = \{1\}$. From the table, we have that non-commuting graph Γ_{D_6} has $\{r, r^2, s, sr, sr^2\}$ as its vertex set. Hence, we can picture Γ_{D_6} as following.

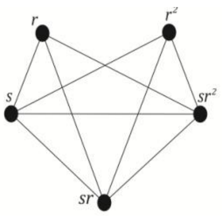

Figure non-commuting graph of D_6

Adjacency matrix for this graph is

$$A(\Gamma_{D_6}) = \begin{array}{c} \\ r \\ r^2 \\ s \\ sr \\ sr^2 \end{array} \begin{array}{c} \begin{array}{ccccc} r & r^2 & s & sr & sr^2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

and its degree matrix is

$$D(\Gamma_{D_6}) = \begin{array}{c} \\ r \\ r^2 \\ s \\ sr \\ sr^2 \end{array} \begin{array}{c} \begin{array}{ccccc} r & r^2 & s & sr & sr^2 \end{array} \\ \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \end{array}$$

So, we have the Laplacian matrix $L(\Gamma_{D_6}) = D(\Gamma_{D_6}) - A(\Gamma_{D_6})$ as follows

$$L(\Gamma_{D_6}) = \begin{array}{c} \\ r \\ r^2 \\ s \\ sr \\ sr^2 \end{array} \begin{array}{c} \begin{array}{ccccc} r & r^2 & s & sr & sr^2 \end{array} \\ \begin{bmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \end{array}$$

Now, we find eigenvalues of $L(\Gamma_{D_6})$ using formula

$$det\left(L(\Gamma_{D_6}) - \lambda I\right) = 0.$$

Because

$$det\left(L(\Gamma_{D_6}) - \lambda I\right) = (-1)(3-\lambda)(-5+\lambda)^2(-5\lambda+\lambda^2)$$

and $det\left(L(\Gamma_{D_6}) - \lambda I\right)$ must equal to zero, we have

$$\lambda_1 = 5,\ \lambda_2 = 3,\ \lambda_3 = 0$$

as eigenvalues for $L(\Gamma_{D_6})$. Finally, we determine the algebraic multiplicity of each eigenvalue and we will have spectrum of the Laplacian matrix of non-commuting graph of dihedral group D_6 as follows

$$Spec_L(\Gamma_{D_6}) = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 1 & 1 \end{bmatrix}.$$

By similar manner we have

$$Spec_L(\Gamma_{D_{10}}) = \begin{bmatrix} 9 & 5 & 0 \\ 5 & 3 & 1 \end{bmatrix}$$

and

$$Spec_L(\Gamma_{D_{14}}) = \begin{bmatrix} 13 & 7 & 0 \\ 7 & 5 & 1 \end{bmatrix}.$$

So, we will have final results for this investigation as the following.

Lemma 1.
Let D_{2n} be dihedral group order $2n$, where n is odd natural numbers and $n \geq 3$. Characteristic polynomial of the Laplacian matrix of non-commuting graph of D_{2n} is

$$p(\lambda) = (-1)(n-\lambda)^{n-2}\left((-2n+1)+\lambda\right)^{n-1}\left((-2n+1)\lambda+\lambda^2\right).$$

and the roots of $p(\lambda) = 0$ are 1, n, and $2n - 1$.

Proof.
Adjacency matrix of $\Gamma_{D_{2n}}$ is $(2n-1 \times 2n-1)$-square matrix

$$A(\Gamma_{D_{2n}}) = \begin{array}{c} \\ n-1\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad}^{n-1} & \overbrace{\qquad\qquad}^{n} \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \vdots \\ 1 & \cdots & 1 & \ddots & 1 & \vdots & 1 \\ \vdots & \ddots & \cdots & \ddots & 0 & \ddots & \vdots \\ 1 & \cdots & 1 & \vdots & 1 & \ddots & 1 \\ \cdots & 1 & \cdots & \ddots & 1 & \cdots & 0 \end{bmatrix} \end{array} \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} 2n-2$$

and its degree matrix is $(2n-1 \times 2n-1)$-square matrix

$$D(\Gamma_{D_{2n}}) = \begin{bmatrix} n & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & n & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 2n-2 & 0 & \vdots & 0 \\ \vdots & 0 & \cdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & 0 & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 2n-2 \end{bmatrix}$$

So, its Laplacian matrix is $L(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) - A(\Gamma_{D_{2n}})$

$$L(\Gamma_{\mathrm{D}_{2n}}) = \begin{array}{c} \\ n-2\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ n\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array}\begin{bmatrix} n & \overbrace{\cdots \quad 0}^{n-2} & \overbrace{-1 \quad \cdots \quad -1 \quad -1}^{n} \\ \vdots & \ddots \quad \vdots & \vdots \quad \ddots \quad \vdots \quad \vdots \\ 0 & \cdots \quad n & -1 \quad \cdots \quad -1 \quad \vdots \\ -1 & \cdots \quad -1 & 2n-2 \quad -1 \quad \vdots \quad -1 \\ \vdots & \ddots \quad \cdots & \ddots \quad \ddots \quad \ddots \quad \vdots \\ -1 & \cdots \quad -1 & \vdots \quad -1 \quad \ddots \quad -1 \\ \cdots & -1 \quad \cdots & \ddots \quad -1 \quad \cdots \quad 2n-2 \end{bmatrix}_{2n-1\times 2n-1} \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} 2n-2$$

We know that characteristic polynomial of $L(\Gamma_{\mathrm{D}_{2n}})$ is

$$det(L(\Gamma_{\mathrm{D}_{14}}) - \lambda I)$$

Aplly the Gaussian elimination procedure we obtained this upper trianggular matrix

$$\begin{vmatrix} (-1) & 0 & 0 & (n-1)-\lambda & \cdots & -1 & -1 \\ \vdots & (n-\lambda) & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & -1 & \cdots & -1 & \vdots \\ 0 & 0 & 0 & (-2n+1)+\lambda & -(-2n+1)+\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & (-2n+1)\lambda+\lambda^2 \end{vmatrix}.$$

Because determinant for this matrix is multiplication of entry in main diagonal, so we obtained

$$p(\lambda) = (-1)(n-\lambda)^{n-2}\left((-2n+1)+\lambda\right)^{n-1}((-2n+1)\lambda+\lambda^2)$$

And, for $p(\lambda) = 0$ we obtained $\lambda_1 = 2n - 1$, $\lambda_2 = n$, and $\lambda_3 = 0$ as the roots of $p(\lambda) = 0$.

Theorem 1.
Let D_{2n} be dihedral group order $2n$, where n is odd natural numbers and $n \geq 3$. Spectrum of the Laplacian matrix of non-commuting graph of D_{2n} is

$$Spec_L(\Gamma_{\mathrm{D}_{2n}}) = \begin{bmatrix} 2n-1 & n & 0 \\ n & n-2 & 1 \end{bmatrix}$$

Proof.
From Lemma 1 we have characteristic polynomial of $L(\Gamma_{\mathrm{D}_{2n}})$ is

$$p(\lambda) = (-1)(n-\lambda)^{n-2}\left((-2n+1)+\lambda\right)^{n-1}((-2n+1)\lambda+\lambda^2).$$

and the roots of $p(\lambda) = 0$ are 1, n, and $2n - 1$. So, we obtained $\lambda_1 = 2n - 1$, $\lambda_2 = n$, and $\lambda_3 = 0$ as eigenvalues of $L(\Gamma_{\mathrm{D}_{2n}})$.

Now, we will determine multiplicities for each eigenvalues. Because the multiplicities are equal to number of basis for space of eigenvectors corresponding to λ_i, $i = 1, 2, 3$, we substitute λ_i to $L(\Gamma_{\mathrm{D}_{2n}}) - \lambda_i I$ and apply Gauss-Jordan elimination to this matrix to obtain row-reduced eselon matrix and then see the number of zero rows of it.

For $\lambda_1 = 2n - 1$, after we eliminate $L(\Gamma_{\mathrm{D}_{2n}}) - \lambda_1 I$, we obtained row-reduced eselon matrix with n zero rows. So, for $\lambda_1 = 2n - 1$ we have its algebraic multiplicity is n.

For $\lambda_2 = n$, after we eliminate $L(\Gamma_{\mathrm{D}_{2n}}) - \lambda_2 I$, we obtained row-reduced eselon matrix with $(n - 2)$ zero rows. So, for $\lambda_1 = 2n - 1$ we have its algebraic multiplicity is $(n - 2)$.

For $\lambda_3 = 0$, after we eliminate $L(\Gamma_{\mathrm{D}_{2n}}) - \lambda_3 I$, we obtained row-reduced eselon matrix with 1 zero row. So, for $\lambda_3 = 0$ we have its algebraic multiplicity is 1.

We conclude that

$$Spec_L(\Gamma_{\mathrm{D}_{2n}}) = \begin{bmatrix} 2n-1 & n & 0 \\ n & n-2 & 1 \end{bmatrix}.$$

CONCLUSION

According to the result, we just determine spectrum of the Laplacian matrix of non-commuting graph of dihedral group D_{2n}, where n is odd natural numbers and $n \geq 3$. We have

$$Spec_L(\Gamma_{\mathrm{D}_{2n}}) = \begin{bmatrix} 2n-1 & n & 0 \\ n & n-2 & 1 \end{bmatrix}.$$

So, we suggest to the reader to investigate spectrum of the Laplacian matrix of non-commuting graph of dihedral group D_{2n}, where n is even natural numbers and $n \geq 3$. Investigation can also be done for spectrum of the Signless Laplacian matrix or detour matrix of non-commuting graph of dihedral group D_{2n}. Similar research can be conducted for symmetric group S_n.

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