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On the Forgotten Index and Jacobson Graphs Associated with Integer Rings Modulo n

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Abstract

This paper investigates the connections between the Jacobson graph and the algebraic properties of rings through the analysis of the Jacobson graph of the ring $\mathbb{Z}_{3p}$, where p is a prime number greater than 3. The Jacobson graph of a commutative ring R is constructed by taking the elements of R, excluding its Jacobson Radical, as vertices, and connecting two distinct vertices if 1 minus their product is not a unit in R. The F-Index is utilized to capture and represent the structural properties of the ring through its associated graph. A detailed examination of the Jacobson Radical, maximal ideals, and vertex degrees in $\mathbb{Z}_{3p}$ leads to the calculation of the F-Index, providing insights into the graph's connectivity and underlying algebraic structure. This study contributes to the intersection of algebra and graph theory, offering a foundation for further research into more complex algebraic structures.

Keywords: F-Index, Jacobson Graph, Jacobson Radical, Maximal ideal, Modulo Ring.
MSC2020: 05C99

Abstrak

Penelitian ini membahas hubungan antara aljabar dan teori graf melalui analisis graf Jacobson dari gelanggang $\mathbb{Z}_{3p}$, di mana p adalah bilangan prima yang lebih besar dari 3. Graf Jacobson dari suatu gelanggang komutatif R dibangun dari elemen-elemen R, kecuali Radikal Jacobson-nya, sebagai titik, dan dua unsur di R yang berbeda akan bertetangga jika 1 dikurangi hasil kali kedua unsur tersebut bukan merupakan unit di R. Pada penelitian ini F-Index digunakan untuk merepresentasikan sifat struktural gelanggang melalui graf Jacobson. Pembahasan mendetail terhadap Radikal Jacobson, ideal maksimal, dan derajat titik dalam $\mathbb{Z}_{3p}$ mengarah pada perhitungan F-Index, yang memberikan wawasan tentang konektivitas graf serta struktur aljabar yang mendasarinya. Studi ini memberikan kontribusi pada persinggungan antara aljabar dan teori graf, serta menyediakan landasan untuk penelitian lebih lanjut mengenai struktur aljabar yang lebih kompleks.

Kata kunci: F-Index, Graf Jacobson, Gelanggang Modulo, Ideal Maksimal, Radikal Jacobson.
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Introduction

Graph theory has become an essential area of mathematics, with significant implications in various fields, including computer science, chemistry, and operations research. Recent studies have highlighted its applications in chemical graph theory, where molecular structures are represented as graphs, facilitating the analysis of chemical properties through topological indices. The exploration of graph properties, such as domination indices, has also gained traction, providing insights into the relationships between different graph characteristics [1], [2].

In the context of algebraic structures, particularly rings, the study of graphs such as zero-divisor graphs and Jacobson graphs has proven valuable. These graphs help elucidate the properties of commutative rings and their zero-divisors, with potential applications in algebraic geometry and coding theory [3]. The interplay between graph theory and algebraic structures is further exemplified in the examination of topological indices related to various graph operations, which can yield insights into molecular graphs and their properties [4]. Moreover, the integration of graph theory with machine learning and data analysis has opened new avenues for research. For instance, graph neural networks (GNNs) have been employed to enhance the performance of various graph analysis tasks, showcasing the versatility of graph theory in modern computational applications [5]. The application of graph theory in decision-making processes, particularly in fuzzy logic and optimization problems, has also been explored, demonstrating its relevance in practical scenarios [6], [7].

The Jacobson graph of a commutative ring is a crucial concept in algebraic graph theory, providing a framework for understanding the relationships between elements of the ring. This graph, denoted as $\mathfrak{J}(R)$, is constructed by considering the elements of a commutative ring R with a nonzero identity, specifically excluding the Jacobson radical $J(R)$. The vertices of $\mathfrak{J}(R)$ are thus the elements of $R \setminus J(R)$, and two distinct vertices x and y are adjacent if and only if $1 - xy$ is not a unit in R [8], [9]. This adjacency condition captures the algebraic relationships inherent in the ring, reflecting the non-invertibility of the product of the vertices.

The significance of the Jacobson graph extends beyond its definition; it serves as a powerful tool for analyzing the structure of the ring R. By examining properties such as connectivity, diameter, and girth, researchers can gain insights into the underlying characteristics of the ring itself [10], [11]. For example, the connectivity of the Jacobson graph can indicate the presence of certain algebraic structures within the ring, while the diameter can provide information about the relationships between its elements. Recent studies have highlighted various aspects of the Jacobson graph, including its applications in understanding the behavior of rings under different operations and its role in characterizing specific types of rings, such as finite commutative rings [12], [13]. Furthermore, the exploration of the Jacobson graph has implications for other areas of mathematics, including representation theory and coding theory, where the properties of the graph can influence the design and analysis of codes [14], [15].

Topological indices are indeed pivotal in chemical graph theory, serving as essential tools for predicting various molecular properties, including stability, toxicity, and reactivity. These indices are numerical values derived from the graph representation of molecules, where atoms are represented as vertices and bonds as edges. The significance of topological indices extends across various domains, particularly in medicinal chemistry and material science, where they facilitate the analysis and understanding of molecular structures[8], [9]. In the context of drug design and development, topological indices are integrated into quantitative structure-activity relationship (QSAR) models. These models leverage the relationships between chemical structure and biological activity to predict the efficacy and toxicity of compounds, thereby streamlining the drug discovery process [10], [11]. The incorporation of topological indices into machine learning frameworks has further enhanced the predictive capabilities of these models, allowing for more accurate simulations of molecular dynamics and interactions [12], [13].

Recent studies have demonstrated the effectiveness of topological indices in various applications, including the prediction of molecular properties relevant to environmental science and toxicology. For instance, researchers have utilized topological indices to assess the environmental impact of chemical compounds and their potential toxicity to living organisms [14], [15]. The versatility of topological indices makes them invaluable in the ongoing quest to design safer and more effective pharmaceuticals and materials.

The Forgotten Index, commonly referred to as the F-index, has emerged as a significant topological index in mathematical chemistry and graph theory. Defined as the sum of the cubes of the vertex degrees in a graph, the F-index is expressed mathematically as:

$$F(G) = \sum_{v \in V(G)} \deg(v)^3$$

where $V(G)$ is the set of vertices in graph G and $\deg(v)$ denotes the degree of vertex v. This index provides valuable insights into the structural characteristics of chemical compounds, enhancing the predictive capabilities of various molecular descriptors [8], [9].

The relevance of the F-index extends beyond traditional molecular graphs; it can also be applied to graphs generated from algebraic structures such as rings. In this context, the F-index can be utilized to analyze the algebraic properties of these structures, offering a deeper understanding of their connectivity and the relationships among their elements [10], [11]. For instance, the F-index can help elucidate the nature of zero-divisor graphs and Jacobson graphs associated with rings, providing insights into the underlying algebraic relationships that govern these structures [12], [13].

Recent studies have highlighted the utility of the F-index in various applications, including the prediction of molecular properties such as stability, reactivity, and biological activity. By incorporating the F-index into quantitative structure-activity relationship (QSAR) models, researchers can improve the accuracy of predictions related to compound efficacy and toxicity [14], [15].

The F-index not only aids in exploring chemical properties but also serves as a bridge between algebraic structures and graph theory. Its application extends to the study of zero-divisor graphs and other ring-related graphs, facilitating a deeper investigation into the algebraic relationships that govern these structures [10]. This dual functionality underscores the significance of the F-index in advancing the field of mathematical chemistry and graph theory.

This research aims to explore and compute the F-index of the Jacobson graph of $\mathbb{Z}_{3p}$, where p is a prime number and $p > 3$. By analyzing the structural properties of this specific Jacobson graph, we seek to contribute to the understanding of the interplay between algebraic structures and graph theory, thereby enhancing the application of the F-index in both mathematical and chemical contexts.

Methods

Theoretical Approach

This research employs a theoretical approach grounded in Graph Theory and Abstract Algebra. It delves into the intrinsic properties of $\mathbb{Z}_{3p}$, for focusing on the ideal structure and the Jacobson Radical, which is subsequently translated into the Jacobson graph structure. The F-Index computation is based on the vertex degrees in this graph, calculated using modular algebraic formulas. Key steps in this approach include determining its Jacobson Radical and maximal ideals, constructing the corresponding Jacobson graph from the remaining elements, calculating the degree of each vertex, and finally, applying the algebraic formula to compute the F-Index from these degrees. This method ensures a comprehensive analysis of the graph's topological index in relation to the algebraic properties of the ring.

Combinatorics and Modular Algebra Approach

In this research, we employ a combinatorics and modular algebra approach to determine the connections between vertices in the Jacobson graph, using modular congruence calculations. This method focuses on computing the greatest common divisor between elements of the ring and its modulus to identify whether two vertices are adjacent. Specifically, adjacency is defined based on whether certain congruence relations hold, utilizing the properties of $\mathbb{Z}_{3p}$. The main steps in this approach include applying modular algebra techniques to systematically determine vertex pairs that are directly connected in the Jacobson graph. Additionally, number theory is used to solve congruence equations, which form the basis for identifying the adjacency relations between vertices.

Proof of Theorems and Lemmas

Finally, a key component of this study involves proving several theorems and lemmas, which form the theoretical foundation for calculating the F-Index of the Jacobson graph generated from $\mathbb{Z}_{3p}$. The proofs of these lemmas are essential, as they establish the structure of maximal ideals, the set of units, and the Jacobson Radical, all of which are critical for constructing the graph and calculating its F-Index. The primary steps in this method include systematically presenting proofs for each relevant theorem and lemma, such as Lemma 1 through 7, which demonstrate the properties of the Jacobson Radical and the maximal ideals of $\mathbb{Z}_{3p}$. These lemmas directly contribute to the formulation of the F-Index, ensuring the accuracy and rigor of the overall calculations.

Results and Discussion

As an illustrative example, we will take $p = 3$ to demonstrate the process. This will involve identifying the elements of $\mathbb{Z}_{3p}$, determining its ideals and maximal ideals, and calculating the Jacobson Radical. Using this information, the Jacobson graph will be constructed, with its vertex set consisting of the elements of $\mathbb{Z}_{3p}$ excluding the Jacobson Radical. After constructing the graph, the degree of each vertex will be calculated, and the F-Index will be determined based on these degrees. This example will serve as a concrete foundation for the formal proofs of the lemmas and theorems that follow.

The set of elements in the ring of integers modulo 15 is given by

$$\mathbb{Z}_{15} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, \overline{11}, \overline{12}, \overline{13}, \overline{14}\}.$$

To determine the Jacobson Radical of $\mathbb{Z}_{15}$, we first need to identify its maximal ideals. The complete set of ideals of $\mathbb{Z}_{15}$ is as follows:

$\langle \bar{0} \rangle = \{\bar{0}\}$

$\langle \bar{1} \rangle = \langle \bar{2} \rangle = \langle \bar{4} \rangle = \langle \bar{7} \rangle = \langle \bar{8} \rangle = \langle \overline{11} \rangle = \langle \overline{12} \rangle = \langle \overline{13} \rangle = \langle \overline{14} \rangle = Z_{15}$

$\langle \bar{3} \rangle = \langle \bar{6} \rangle = \langle \bar{9} \rangle = \langle \overline{12} \rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}\}$

$\langle \bar{5} \rangle = \{\bar{0}, \bar{5}, \overline{10}\}$

Thus, the maximal ideals of $\mathbb{Z}_{15}$ are $\langle 3 \rangle = \{0, 3, 6, 9, 12\}$ and $\langle 5 \rangle = \{0, 5, 10\}$, as these ideals are not contained within any other proper ideal. Therefore, the Jacobson Radical of $\mathbb{Z}_{15}$ is

$$J(Z_{15}) = \{0, 3, 6, 9, 12\} \cap \{0, 5, 10\} = \{0\}.$$

Based on the definition of Jacobson graph, the set of nodes of $\mathfrak{J}_{\mathbb{Z}_{15}}$ is

$$\begin{aligned} V(\mathfrak{J}_{\mathbb{Z}_{15}}) &= \mathbb{Z}_{15} - J(Z_{15}) \\ &= \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, 11, 12, 13, \overline{14}\}. \end{aligned}$$

and the unit of the ring $\mathbb{Z}_{15}$ is $U(\mathbb{Z}_{15}) = \{\bar{1}, \bar{2}, \bar{4}, \bar{7}, \bar{8}, \overline{11}, \overline{13}, \overline{14}\}$. Then, we will find the vertices that are adjacent using the following Table 1:

Table 1. $1 - \overline{xy}$ for $x, y \in \mathbb{Z}_{15} \backslash J(\mathbb{Z}_{15})$

x \ y	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{13}$	$\overline{14}$
$\bar{1}$	$\bar{0}$	$\overline{14}$	$\overline{13}$	$\overline{12}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$
$\bar{2}$	$\overline{14}$	$\overline{12}$	$\overline{10}$	$\bar{8}$	$\bar{6}$	$\bar{4}$	$\bar{2}$	$\bar{0}$	$\overline{13}$	$\overline{11}$	$\bar{9}$	$\bar{7}$	$\bar{5}$	$\bar{3}$
$\bar{3}$	$\overline{13}$	$\overline{10}$	$\bar{7}$	$\bar{4}$	$\bar{1}$	$\overline{13}$	$\overline{10}$	$\bar{7}$	$\bar{4}$	$\bar{1}$	$\overline{13}$	$\overline{10}$	$\bar{7}$	$\bar{4}$
$\bar{4}$	$\overline{12}$	$\bar{8}$	$\bar{4}$	$\bar{0}$	$\overline{11}$	$\bar{7}$	$\bar{3}$	$\overline{14}$	$\overline{10}$	$\bar{6}$	$\bar{2}$	$\overline{13}$	$\bar{9}$	$\bar{5}$
$\bar{5}$	$\overline{11}$	$\bar{6}$	$\bar{1}$	$\overline{11}$	$\bar{6}$	$\bar{1}$	$\overline{11}$	$\bar{6}$	$\bar{1}$	$\overline{11}$	$\bar{6}$	$\bar{1}$	$\overline{11}$	$\bar{6}$
$\bar{6}$	$\overline{10}$	$\bar{4}$	$\overline{13}$	$\bar{7}$	$\bar{1}$	$\overline{10}$	$\bar{4}$	$\overline{13}$	$\bar{7}$	$\bar{1}$	$\overline{10}$	$\bar{4}$	$\overline{13}$	$\bar{7}$
$\bar{7}$	$\bar{9}$	$\bar{2}$	$\overline{10}$	$\bar{3}$	$\overline{11}$	$\bar{4}$	$\overline{12}$	$\bar{5}$	$\overline{13}$	$\bar{6}$	$\overline{14}$	$\bar{7}$	$\bar{0}$	$\bar{8}$
$\bar{8}$	$\bar{8}$	$\bar{0}$	$\bar{7}$	$\overline{14}$	$\bar{6}$	$\overline{13}$	$\bar{5}$	$\overline{12}$	$\bar{4}$	$\overline{11}$	$\bar{3}$	$\overline{10}$	$\bar{2}$	$\bar{9}$
$\bar{9}$	$\bar{7}$	$\overline{13}$	$\bar{4}$	$\overline{10}$	$\bar{1}$	$\bar{7}$	$\overline{13}$	$\bar{4}$	$\overline{10}$	$\bar{1}$	$\bar{7}$	$\overline{13}$	$\bar{4}$	$\overline{10}$
$\overline{10}$	$\bar{6}$	$\overline{11}$	$\bar{1}$	$\bar{6}$	$\overline{11}$	$\bar{1}$	$\bar{6}$	$\overline{11}$	$\bar{1}$	$\bar{6}$	$\overline{11}$	$\bar{1}$	$\bar{6}$	$\overline{11}$
$\overline{11}$	$\bar{5}$	$\bar{9}$	$\overline{13}$	$\bar{2}$	$\bar{6}$	$\overline{10}$	$\overline{14}$	$\bar{3}$	$\bar{7}$	$\overline{11}$	$\bar{0}$	$\bar{4}$	$\bar{8}$	$\overline{12}$
$\overline{12}$	$\bar{4}$	$\bar{7}$	$\overline{10}$	$\overline{13}$	$\bar{1}$	$\bar{4}$	$\bar{7}$	$\overline{10}$	$\overline{13}$	$\bar{1}$	$\bar{4}$	$\bar{7}$	$\overline{10}$	$\overline{13}$
$\overline{13}$	$\bar{3}$	$\bar{5}$	$\bar{7}$	$\bar{9}$	$\overline{11}$	$\overline{13}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{6}$	$\bar{8}$	$\overline{10}$	$\overline{12}$	$\overline{14}$
$\overline{14}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\bar{0}$

From the table above, the set of edges on the $\mathfrak{J}_{\mathbb{Z}_{15}}$ is

$$E(\mathfrak{J}_{\mathbb{Z}_{15}}) = \{(1,4), (1,6), (1,7), (1,10), (1,11), (1,13), (2,3), (2,5), (2,8), (2,11), (2,13), (2,14), (3,7), (3,12), (4,7), (4,9), (4,10), (4,13), (4,14), (5,8), (5,11), (5,14), (2,14), (3,7), (3,12), (4,7), (4,9), (4,10), (4,13), (4,14), (5,8), (5,11), (5,14), (6,11), (7,8), (7,10), (7,13), (8,11), (8,12), (8,14), (9,14), (10,13), (11,14), (12,13)\}.$$

$\mathfrak{J}_{\mathbb{Z}_{15}}$ is shown by Figure 1.

Based on Figure 4.1, it can be seen that the degree of point on $\mathfrak{J}_{\mathbb{Z}_{15}}$ is as follows Table 2:

Table 2. The vertex degrees of $\mathfrak{J}_{\mathbb{Z}_{15}}$

u	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{13}$	$\overline{14}$
$\deg(u)$	6	6	3	6	4	2	6	6	2	4	6	3	6	6

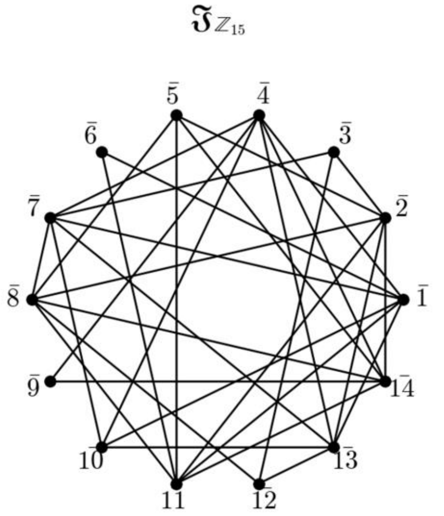

Figure 1 $\mathfrak{I}_{\mathbb{Z}_{15}}$

Therefore, the *F-Index* of $\mathfrak{I}_{\mathbb{Z}_{15}}$ is

$$F\left(\mathfrak{I}_{Z_{15}}\right) = \sum_{u \in V(\mathfrak{I}_{Z_{15}})} \deg(u)^3$$

$$\begin{aligned}
&= \deg(\bar{1})^3 + \deg(\bar{2})^3 + \deg(\bar{3})^3 + \deg(\bar{4})^3 + \deg\left(\bar{5}\right)^3 + \deg(\bar{6})^3 + \deg(\bar{7})^3 + \\
&\quad \deg(\bar{8})^3 + \deg(\bar{9})^3 + \deg(\overline{10})^3 + \deg(\overline{11})^3 + \deg(\overline{12})^3 + \deg(\overline{13})^3 + \\
&\quad \deg(\overline{14})^3 + \deg\left(\overline{15}\right)^3 \\
&= 6^3 + 6^3 + 3^3 + 6^3 + 4^3 + 2^3 + 6^3 + 6^3 + 2^3 + 4^3 + 6^3 + 3^3 + 6^3 + 6^3 \\
&= 8 \cdot 6^3 + 2 \cdot 3^3 + 2 \cdot 4^3 + 2 \cdot 2^3 \\
&= 8 \cdot 6^3 + 2 \cdot 3^3 + 2 \cdot 4^3 + 2^4
\end{aligned}$$

Following the visualization of $\mathfrak{I}_{\mathbb{Z}_{15}}$, the subsequent objective is to generalize these results for any prime number greater than 3. This extension is crucial for establishing a broader theoretical framework applicable to a wider class of prime numbers. Prior to proving the main theorem, however, it is necessary to first establish a series of foundational results. Specifically, we will construct and rigorously prove seven lemmas, each of which addresses key properties and structural elements required for the final proof. These lemmas serve as essential prerequisites, ensuring the mathematical rigor and completeness of the argument before advancing to the formal proof of the main theorem.

Lemma 1. Let p be a prime number, $p \geq 5$. Then $\mathbb{Z}_{3p}$ has exactly two distinct non-zero proper ideals, namely $\langle \bar{3} \rangle$ and $\langle \bar{p} \rangle$. Hence $\langle \bar{3} \rangle$ and $\langle \bar{p} \rangle$ are maximal ideals in $\mathbb{Z}_{3p}$.

Proof.

Suppose $\bar{x} \in \mathbb{Z}_{3p}$, then there are four cases to prove that $\langle \bar{3} \rangle$ and $\langle \bar{p} \rangle$ are maximal ideals in $\mathbb{Z}_{3p}$, which are as follows:

Case 1. $x = \overline{3k}$ for $k = \{1,2, \dots, p-1\}$.

It will be shown that $\langle x \rangle = \langle \bar{3} \rangle$. Since $x = \overline{3k} = \bar{3} \cdot \bar{k}$ then $\bar{x} \in \langle \bar{3} \rangle$, consequently $\langle \bar{x} \rangle \subseteq \langle \bar{3} \rangle$. Conversely, it will be shown that $\langle \bar{3} \rangle \subseteq \langle x \rangle$. Note that:

$$xa \equiv 3(mod\ 3p) \Leftrightarrow (3k)a \equiv 3(mod\ 3p)$$

has a solution because GCD$(3k, 3p)$ divides 3. That is, there is $a \in \mathbb{Z}_{3p}$ such that $\bar{3} \in \langle \bar{x} \rangle$. Therefore, $\langle \bar{3} \rangle \subseteq \langle x \rangle$ and since $\langle \bar{x} \rangle \subseteq \langle \bar{3} \rangle$ and $\langle \bar{3} \rangle \subseteq \langle \bar{x} \rangle$, we conclude that $\langle \bar{x} \rangle = \langle \bar{3} \rangle$.

Case 2 $x = \overline{3k-1}$ where $k = \{1,2,\dots,p\}$.

Note that $x \neq p$, so that GCD $(x,p) = 1$. Thus $\bar{x}$ is the unit in $\mathbb{Z}_{3p}$, therefore $\langle x \rangle = \mathbb{Z}_{3p}$.

Case 3 $x = \bar{0}$.

Since $x = \bar{0}$ it is obvious that $\langle \bar{x} \rangle = \langle \bar{0} \rangle$.

Case 4 $x = \bar{p}$.

Since p is a prime number and $p \neq 3$ it follows that $3 \nmid p$, hence $\langle \bar{p} \rangle \notin \langle \bar{3} \rangle$. The equivalence relation $3a \equiv p(mod\ 3p)$ has no solution because $gcd(3,3p) = 3$, and since $3 \nmid p$, this implies that $\bar{3} \notin \langle \bar{p} \rangle$, therefore $\langle \bar{3} \rangle \nsubseteq \langle \bar{p} \rangle$. It will then be shown that $\langle \bar{p} \rangle \neq \mathbb{Z}_{3p}$. The equivalence relation $pa \equiv 1(mod\ 3p)$ has no solution because $gcd(p,3p) = p$, and since $p \nmid 1$, it follows that $\bar{1} \notin \langle \bar{p} \rangle$. Thus, $\langle \bar{p} \rangle \neq \mathbb{Z}_{3p}$ and $\langle \bar{p} \rangle$ is a proper ideal of $\mathbb{Z}_{3p}$.

Therefore, $\langle \bar{3} \rangle$ and $\langle \bar{p} \rangle$ are the maximal ideals of $\mathbb{Z}_{3p}$. ■

With the commencement of Lemma 1, we have confirmed the existence of exactly two distinct non-zero proper ideals in $\mathbb{Z}_{3p}$, namely $\langle \bar{3} \rangle$ and $\langle \bar{p} \rangle$, thus establishing their status as maximal ideals. This result provides a foundational understanding of the ideal structure of the ring. Building upon this, we now shift our focus to a more detailed exploration of the unit group of $\mathbb{Z}_{3p}$, which plays a crucial role in the ring's algebraic properties. In Lemma 2, we will characterize the set of units, determining which elements in $\mathbb{Z}_{3p}$ are invertible, and further investigate the relationship between these units and the ring's structure.

Lemma 2. Suppose that p is a prime number and $p > 3$. The unit set of the ring $\mathbb{Z}_{3p}$ is

$$U(\mathbb{Z}_{3p}) = \{x \in \mathbb{Z}_{3p} | x \notin \{\bar{0}, \bar{p}, \overline{2p}\} \cup \{\overline{3n} | n = 1,2,\dots,p-1\}\}.$$

Proof:

Note that the elements of $\mathbb{Z}_{3p}$ is $\{0,1,2,\dots,3p-1\}$. If $x = \bar{0}$, $\bar{0} \cdot y = y \cdot \bar{0} = \bar{0} \neq \bar{1}, \forall y \in \mathbb{Z}_{3p}$. Thus, $\bar{0} \notin U(\mathbb{Z}_{3p})$. If $x \neq \bar{0}$, $x \in U(\mathbb{Z}_{3p})$ if and only if there exists $y \in \mathbb{Z}_{3p}$ such that $xy = 1$, in other words $xy \equiv 1(mod\ 3p)$ has a solution. Accordingly, there exists $r \in \mathbb{Z}$ such that

$$xy \equiv 1(mod\ 3p) \Leftrightarrow xy - 1 = (3p)r \Leftrightarrow xy - (3p)r = 1.$$

Suppose that $t \in \mathbb{Z}$ and $t = gcd(x,3p)$. We obtain

$$t|x \text{ and } t|3p \Leftrightarrow t|xy - (3p)s \Leftrightarrow t|1.$$

There exists $k \in \mathbb{Z}$ such that $1 = tk \Leftrightarrow t = 1$ or $t = -1$. For $t = 1$, it is obtained that

$$gcd(x,3p) = 1.$$

Therefore, $x \in U(\mathbb{Z}_{3p})$ is only satisfied if and only if $gcd(x,3p) = 1$. If $x = \bar{p}$, then $gcd(x,3p) = p \neq 1$, therefore $x \notin U(\mathbb{Z}_{3p})$. If $x = \overline{2p}$, then $gcd(x,3p) = p \neq 1$ and implies that $x \notin U(\mathbb{Z}_{3p})$. If $x = \overline{3n}; n \in \{1,2,\dots,p-1\}$, then $gcd(x,3p) = p \neq 1$ and $x \notin U(\mathbb{Z}_{3p})$.

The factors of $3p$ are $1, 3,$ and p. If $x \in \mathbb{Z}_{3p}$ where $x \notin \{\overline{3n}; n = 1,2,\dots,p-1\} \cup \{\bar{0}, \bar{p}, \overline{2p}\}$, then $gcd(x,3p) = 1$. Therefore $gcd(x,3p) = 1, \forall x \notin \{\overline{3n}; n = 1,2,\dots,p-1\} \cup \{\bar{0}, \bar{p}, \overline{2p}\}$ and it is proven that $U(\mathbb{Z}_{3p}) = \{x \in \mathbb{Z}_{3p} | x \notin \{\bar{0}, \bar{p}, \overline{2p}, \overline{3n}; n = 1,2,\dots,p-1\}\}$.

Having characterized the set of units of $\mathbb{Z}_{3p}$ In Lemma 2, where we identified which elements of the ring are invertible, we now proceed to examine another fundamental aspect of the ring's structure: its Jacobson Radical. In Lemma 3, we will demonstrate that the Jacobson Radical of $\mathbb{Z}_{3p}$ consists solely of the element 0, highlighting the simplicity of the radical in this specific context and its implications for the overall structure of the ring.

Lemma 3. Suppose p is a prime number and $p > 3$. The Jacobson Radical of $\mathbb{Z}_{3p}$ is $\{\bar{0}\}$.
Proof:
Take $x \in (\langle\bar{3}\rangle \cap \langle\bar{p}\rangle) = J(\mathbb{Z}_{3p})$. Since $\langle\bar{p}\rangle = \{\bar{0}, \bar{p}\}$ then $x = \bar{0}$ or $\bar{x} = \bar{p}$. Note that $\bar{p} \notin \langle\bar{3}\rangle$ because $3a \equiv p(mod\ 3p)$ has no solution, it must be $\bar{x} \in \langle\bar{3}\rangle \cap \langle\bar{p}\rangle$ if and only if $x = \bar{0}$. Thus, $J(\mathbb{Z}_{3p}) = \{\bar{0}\}$.

With the conclusion of Lemma 3, which confirms that the Jacobson Radical of $\mathbb{Z}_{3p}$ is restricted to the element 0, this finding allows us to derive a significant corollary. The corollary builds on this by identifying the vertex set of the Jacobson graph, specifically excluding the Jacobson Radical. This exclusion provides key insights into the structure of the graph, paving the way for further exploration of its properties.

Corollary of Lemma 3

Suppose p is a prime number with $p > 3$. $V\left(\mathfrak{J}_{\mathbb{Z}_{3p}}\right) = \mathbb{Z}_{3p}\backslash\{\bar{0}\}$.
Proof:
It is known that $V\left(\mathfrak{J}_{\mathbb{Z}_{3p}}\right) = \mathbb{Z}_{3p}\backslash J(\mathbb{Z}_{3p})$, and by Lemma 2 it is clear that $V\left(\mathfrak{J}_{\mathbb{Z}_{3p}}\right) = \mathbb{Z}_{3p}\backslash\{\bar{0}\}$.

Following the corollary of Lemma 3, where the vertex set of the Jacobson graph was clarified, we now delve deeper into the structural characteristics of these vertices. Lemma 4 will focus on the degrees of specific vertices within the Jacobson graph, particularly for the elements p and $2p$. By analyzing these vertex degrees, we can further understand the connectivity within the graph and its implications for the broader algebraic structure of $\mathbb{Z}_{3p}$.

To simplify the proof that follows, some adjustments will be made in the proof. Define:

$$V_1 \coloneqq \{p, 2p\},$$
$$V_2 \coloneqq \{3n \mid n = 1, 2, \dots, p-1\},$$
$$V_3 \coloneqq V - (V_1 \cup V_2).$$

Recall that for every $x, y \in V\left(\mathfrak{J}(\mathbb{Z}_{3\mathrm{p}})\right)$, x and y will be adjacent if and only if $1 - xy \notin U(\mathbb{Z}_{3p})$ or in other words when

$$xy \equiv 1(mod\ 3) \quad \text{or} \quad xy \equiv 1(mod\ p).$$

Lemma 4 Suppose p is a prime number with $p > 3$. $\deg_{\mathfrak{J}(\mathbb{Z}_{3\mathrm{p}})}(\bar{p}) = \deg_{\mathfrak{J}(\mathbb{Z}_{3\mathrm{p}})}(\overline{2p}) = p - 1$.
Proof:

Since $gcd(p, p) \neq 1$, then the equivalence relation $xp \equiv 1(\mathrm{mod}\ p)$ has no solution. Since $gcd(p, 3) = 1$, then the equivalence relation $xp \equiv 1(\mathrm{mod}\ 3)$ has a unique solution. Suppose the solution is s, then for some $m \in \{0, 1, 2, \dots, p-1\}$, $s + 3m$ is the solution of the congruence. Thus there are p solutions to the congruence. Since $p^2 \equiv 1(\mathrm{mod}\ 3)$, then from the definition of Jacobson graph there is $p - 1$ vertices that adjacent to $\bar{p}$.

Furthermore, since $gcd(2p, p) \neq 1$, then the equivalence relation $x(2p) \equiv 1(\mathrm{mod}\ p)$ has no solution. Since $gcd(2p, 3) = 1$, then $x(2p) \equiv 1(\mathrm{mod}\ 3)$ has a unique solution. Suppose the solution is s, then for some $m \in \{0, 1, 2, \dots, p-1\}$, $s + 3m$ is the solution of the congruence. Thus there are p solutions to the congruence. Since $4p^2 \equiv 1(\mathrm{mod}\ 3)$, then from the definition of Jacobson graph there is $p - 1$ vertices that adjacent to $\overline{2p}$. ■

With the degrees of vertices p and $2p$ established in Lemma 4, the next series of lemmas will expand our understanding of the Jacobson graph by focusing on different subsets of vertices. Lemma 5 explores the vertices in V_2, demonstrating that each has exactly three distinct adjacency solutions.

Following this, Lemma 6 investigates the specific properties of a subset of V_2, revealing the unique elements that satisfy certain congruence conditions. Finally, Lemma 7 completes the analysis by determining the degrees of vertices in V_3, offering a comprehensive view of the graph's connectivity. These results together provide a detailed and layered understanding of the Jacobson graph and its underlying algebraic structure.

Lemma 5. $1 - xv \notin U(\mathbb{Z}_{3p})$ has exactly three solutions for every $v \in V_2$.
Proof:
Let $v = 3n$ for some $n \in \mathbb{Z}$. Since $gcd(v, 3) \neq 1$, then $xv \equiv 1(\text{mod } 3)$ has no solution. Since $gcd(v, p) = 1$, then $xv \equiv 1(\text{mod } p)$ has a single solution. Suppose the solution is s, then for some $m \in \{0, 1, 2\}$, $s + mp$ is a solution of the congruence. Thus there are three solutions to the congruence. ∎

Lemma 6. Let $U \coloneqq \{v \in V_2 \mid v^2 \equiv 1(\text{mod } p)\}$. Then $|U| = 2$.
Proof:
Before proving the cardinality of U, we will determine the solution of

$$x^2 \equiv 1(mod\ p)$$

for a prime number $p > 3$.
Note that

$$x^2 - 1 \equiv 0(mod\ p) \Leftrightarrow (x + 1)(x - 1) \equiv 0(mod\ p)$$

which means $x + 1 \equiv 0(mod\ p)$ or $x - 1 \equiv 0(mod\ p)$.
From $x + 1 \equiv 0(mod\ p)$,

$$x + 1 \equiv 0(mod\ p) \Rightarrow x \equiv -1(mod\ p) \Rightarrow x \equiv p - 1(mod\ p),$$

and from $x - 1 \equiv 0(mod\ p)$,

$$x - 1 \equiv 0(mod\ p) \Rightarrow x \equiv 1(mod\ p).$$

Since $x \in \mathbb{Z}_{3p}$, from the equivalence $x \equiv p - 1(mod\ p)$, it is obtained that

$$x \in \{p - 1, 2p - 1, 3p - 1\}.$$

Similarly, from $x \equiv 1(mod\ p)$, it is obtained that $x \in \{1, p + 1, 2p + 1\}$.
The next step is to prove that the cardinality of U is $|U| = 2$.
Case 1. If $p \equiv 1(mod\ 3)$, then

$$\begin{aligned} p - 1 &\equiv 0(mod\ 3), \\ 2p - 1 &\equiv 1(mod\ 3), \\ 3p - 1 &\equiv 2(mod\ 3), \\ 1 &\equiv 1(mod\ 3), \\ p + 1 &\equiv 2(mod\ 3), \text{ and} \\ 2p + 1 &\equiv 0(mod\ 3). \end{aligned}$$

Thus, $U = \{p - 1, 2p + 1\}$.

Case 2. If $p \equiv 2(mod\ 3)$then

$$\begin{aligned} p - 1 &\equiv 1(mod\ 3), \\ 2p - 1 &\equiv 0(mod\ 3), \\ 3p - 1 &\equiv 2(mod\ 3), \\ 1 &\equiv 1(mod\ 3), \\ p + 1 &\equiv 0(mod\ 3), \text{ and} \\ 2p + 1 &\equiv 2(mod\ 3). \end{aligned}$$

Thus, $U = \{2p - 1, p + 1\}$. ∎

Corollary of Lemma 6. Let p be a prime number with $p > 3$.

$$\left|\left\{v \in V_2 \middle| \deg_{\mathfrak{I}_{\mathbb{Z}_{3p}}}(v) = 3\right\}\right| = p - 3.$$

and

$$\left|\left\{v \in V_2 | \deg_{\mathfrak{I}_{\mathbb{Z}_{3p}}}(v) = 2\right\}\right| = 2.$$

Lemma 7. Let p be a prime number with $p > 3$.

$$\deg_{\mathfrak{I}_{\mathbb{Z}_{3p}}}(v) = p + 1, \forall v \in V_3.$$

Proof:

$xv \equiv 1(mod\ p)$ has a single solution $x = v^{p-2}$ for every $v \in \mathbb{Z}_{3p} - \{0\}$. Therefore, there are three solutions of the equivalence relation for every $v \in \mathbb{Z}_{3p} - \{0\}$. Let

$$S_p(v) := \{x \in \mathbb{Z}_{3p} : xv \equiv 1(\text{mod } p)\}$$

for every $v \in \mathbb{Z}_{3p}$. Then

$$|S_p(v)| = 3, \forall v \in \mathbb{Z}_{3p} - \{0, p, 2p\}. \tag{1}$$

Accordingly, $yv \equiv 1(mod\ 3)$ has a single solution $y = v$ for every $v \in \mathbb{Z}_{3p} - \{0\}$. Let

$$S_3(v) := \{x \in \mathbb{Z}_{3p} : xv \equiv 1(\text{mod } 3)\}$$

for every $v \in \mathbb{Z}_{3p}$. Then

$$|S_3(v)| = p, \forall v \in \mathbb{Z}_{3p} - \{0,3,6,\ldots,3p-3\}. \tag{2}$$

Next, we will determine $S_p(v) \cap S_3(v), \forall v \in V_3$. Take an arbitrary $v \in V_3$. Recall that if $vx \equiv 1(mod\ 3)$ then $x \equiv 1(mod\ 3)$ or $x \equiv 2(mod\ 3)$, and there is exactly one solution of $x \in \mathbb{Z}_{3p}$. Therefore, $vx \equiv 1(mod\ p)$.

Next, it will be determined x such that x satisfies both congruences

$$vx \equiv 1(mod\ 3)$$

and

$$vx \equiv 1(mod\ p).$$

Since 3 and p are coprime, then there exists an x such that $vx \equiv 1(mod\ 3p)$.

From the above results, it is obtained that

$$|S_p(v) \cap S_3(v)| = 1, \forall v \in V_3. \tag{3}$$

It is straightforward to show that $x^2 \equiv 1(mod\ 3), \forall x \in V_3$, meaning that there is exactly one element $x \in S_3(v)$ such that $x = v$.

From (1), (2), (3), and the fact that every vertex in $\mathfrak{I}_{\mathbb{Z}_{3p}}$ is not adjacent to itself, then

$$\begin{aligned}\deg_{\mathfrak{I}_{\mathbb{Z}_{3p}}}(v) &= |S_3(v)| + |S_p(v)| - |S_p(v) \cap S_3(v)| - 1\\ &= p + 3 - 1 - 1\\ &= p + 1\end{aligned}$$

for every $v \in V_3$.

With the detailed analysis of the Jacobson graph structure through the preceding lemmas, we now possess a thorough understanding of the vertex degrees and the interactions between elements in $\mathbb{Z}_{3p}$. These results lay a strong foundation for proceeding to the core outcome of this study, the calculation of the F-Index of the Jacobson graph. By examining all the essential components, we are now prepared to prove the theorem, which will show how the F-Index connects to the algebraic structure of $\mathbb{Z}_{3p}$ and its topological features.

Theorem 1. The *F-Index* of the Jacobson graph of the ring of integers modulo $3p$, for p is a prime number and $p > 3$ is:

$$F\left(\mathfrak{J}_{\mathbb{Z}_{3p}}\right) = 2 \cdot (p-1)^3 + (p-3) \cdot 3^3 + 2^4 + (2p-2) \cdot (p+1)^3$$

Proof:

$$\begin{aligned}
F\left(\mathfrak{J}_{Z_{3p}}\right) &= \sum_{v \in V\left(\mathfrak{J}_{\mathbb{Z}_{3p}}\right)} \deg(v)^3 \\
&= \sum_{v \in V_1} \deg(v)^3 + \sum_{v \in V_2} \deg(v)^3 + \sum_{v \in V_3} \deg(v)^3 \\
&= 2 \cdot (p-1)^3 + (p-1-2) \cdot 3^3 + 2 \cdot 2^3 + \big((3p-1) - (2) - (p-1-2) - 2\big) \cdot (p+1)^3 \\
&= 2 \cdot (p-1)^3 + (p-3) \cdot 3^3 + 2^4 + (2p-2) \cdot (p+1)^3
\end{aligned}$$

Conclusion

In this study, we have explored the structural properties of the Jacobson graph of the ring $\mathbb{Z}_{3p}$, with a specific focus on computing the F-Index. Through a series of lemmas, we examined key components such as the vertex degrees, the Jacobson Radical, and the maximal ideals within the ring. These findings allowed us to establish a comprehensive understanding of the graph's connectivity and algebraic relationships. The calculation of the F-Index demonstrated how topological indices provide insights into both the algebraic structure of $\mathbb{Z}_{3p}$ and its broader graph-theoretical properties. This work contributes to the ongoing intersection between algebra and graph theory, highlighting the relevance of the F-Index in analyzing rings and their associated graphs. Future research could further investigate the application of this approach to more complex rings and other topological indices, potentially uncovering new connections in both mathematical and chemical contexts.

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