Malaysian Journal of Mathematical Sciences 18(4): 807–819 (2024)
https://doi.org/10.47836/mjms.18.4.08

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

A Bipartite Graph Associated to Elements and Class Equivalences of A Finite Heap

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Received: 28 March 2024
Accepted: 8 August 2024

Abstract

In this study, we introduce a bipartite graph associated to elements and equivalence classes of a heap. We find some invariant numbers of the graph. Furthermore, we associate the graph properties of the graph of domain and codomain of surjective heap morphism. We also investigate the tensor product of the bipartite graph of heaps.

Keywords: bipartite graph; heap; heap morphism; invariant number; tensor product of graph.

1 Introduction

There are many researches which associate some algebraic structures to certain graphs. Groups and rings are the most studied algebraic structures associated to graphs. Mostly, the vertices of the graphs are the elements of those algebraic structures, such as coprime graph [15], commuting graph [17], power graph [7], Cayley graph [13], non-commuting graph [1], generating graph [14], on the non-zero divisor graphs of some finite commutative rings [19], neighbors degree sum energy of commuting and non-commuting graphs for dihedral groups [12]. Furthermore, there are graphs whose vertex sets are the set of subgroups, such as intersection graph [2], co-maximal subgroup graph [10], and order divisor graphs [8]. Assume that G is a finite group. Let $A = G$ and $S = \{X | X \leq G\}$. A bipartite graph associated to elements and cosets of subgroups of G is an undirected graph which is simple. The vertex set of this graph is $A \cup S$ and $g \in A$, $X \in S$ are adjacent whenever $gX = Xg$ [3].

Heap is another algebraic structure equipped with an associative ternary operation which is first introduced by [5, 18] . Unlike groups, heaps do not have a unique identity element. Any element of heap can be an identity element of a group constructed by defining certain binary operation. Therefore, we can consider heaps as groups. In this study, we define a bipartite graph associated to elements and sub-heaps on heaps. We observe the invariant number of the graph, such as diameter and girth. Moreover, if we have a surjective heap morphism between two heaps, we will associate some properties of the graphs of the domain and codomain. The last, we also investigate the tensor product of the bipartite graph of heaps.

2 Materials and Methods

There are two basic concepts which will be used in this study, those are heaps and graphs theory. The concept of heap theory in this section is based on [6, 16], while that of graph is based on [9].

2.1 Heaps and groups

Assume that $\mathcal{H} \neq \emptyset$ and $[-,-,-]$ is a ternary operation on $\mathcal{H}$. The set $\mathcal{H}$ is called a heap whenever it satisfies associativity and Mal'cev identity, those are $\big[[p,q,r],s,t\big] = \big[p,q,[r,s,t]\big]$ and $[p,p,q] = q = [q,p,p]$ for every $p,q,r,s,t \in \mathcal{H}$. The notation $\big(\mathcal{H},[-,-,-]\big)$ or simply $\mathcal{H}$ will denote a heap $\mathcal{H}$ with ternary operation $[-,-,-]$. A heap morphism is a map ϕ from heap $\mathcal{H}$ to heap $\mathcal{H}'$ such that $\phi\big([p,q,r]\big) = \big[\phi(p),\phi(q),\phi(r)\big]$ for every $p,q,r \in \mathcal{H}$.

Let, $\mathcal{S} \neq \emptyset$ and $\mathcal{S} \subseteq \mathcal{H}$. The set $\mathcal{S}$ is called sub-heap if $[p,q,r] \in \mathcal{S}$ for every $p,q,r \in \mathcal{S}$. Furthermore, sub-heap $\mathcal{S}$ of $\mathcal{H}$ is called normal if there exist $x \in \mathcal{S}$ such that for every $h \in \mathcal{H}$, $s \in \mathcal{S}$ satisfying $[s,x,h] = [h,x,z]$ for some $z \in \mathcal{S}$. Note that for any $w \in \mathcal{H}$, the set $\{w\}$ is a normal sub-heap of $\mathcal{H}$. We can make some groups from a heap $\big(\mathcal{H},[-,-,-]\big)$ by taking any element $e \in \mathcal{H}$ to define the following binary operation,

$$p \cdot_e q = [p,e,q]. \tag{1}$$

The set $\mathcal{H}$ with binary operation $\cdot_e$ is a group and e becomes the identity element. Otherwise, if we have a group $(\mathcal{G}, \cdot)$ then it can be constructed a heap $\mathcal{G}$ with the ternary operation,

$$[l, m, n] = lm^{-1}n. \tag{2}$$

Every subgroup $\mathcal{G}'$ of $\mathcal{G}$ is automatically being sub-heap of heap $\mathcal{G}$ since $[l, m, n] = lm^{-1}n \in \mathcal{G}'$ for every $l, m, n \in \mathcal{G}'$. Moreover, a sub-heap $\mathcal{S}$ of $\mathcal{H}$ is subgroup of $(\mathcal{H}, \cdot_\epsilon)$ iff $\epsilon \in \mathcal{S}$.

Note that for every sub-heap $\mathcal{S}$ of $\mathcal{H}$, it can be defined an equivalence relations $\sim_{\mathcal{S}}$ and ${}_{\mathcal{S}}\sim$ on $\mathcal{H}$. For every $x, y \in \mathcal{H}$, the relations are defined respectively as,

$$x \sim_{\mathcal{S}} y \text{ iff } [x, y, s] \in \mathcal{S}, \qquad \text{for some } s \in \mathcal{S}, \tag{3}$$

and

$$x \;{}_{\mathcal{S}}\!\sim y \text{ iff } [t, x, y] \in \mathcal{S}, \qquad \text{for some } t \in \mathcal{S}. \tag{4}$$

The equivalence class of x with respect to relation $\sim_{\mathcal{S}}$ $({}_{\mathcal{S}}\sim)$ is denoted by $\overline{x}_{\mathcal{S}}({}_{\mathcal{S}}\overline{x})$. When $e \in \mathcal{S}$, these equivalence classes associate with left and right cosets of subgroup $\mathcal{S}$ of group $(\mathcal{H}, \cdot_e)$. To be specific, $\overline{x}_{\mathcal{S}} = \mathcal{S}x$ and ${}_{\mathcal{S}}\overline{x} = x\mathcal{S}$. Otherwise, if we have a subgroup $\mathcal{G}'$ of group $\mathcal{G}$, then it satisfies that $\overline{y}_{\mathcal{G}'} = \mathcal{G}'y$ and ${}_{\mathcal{G}'}\overline{y} = y\mathcal{G}'$ for every $y \in \mathcal{G}$. Hence, $\mathcal{S}$ is a normal sub-heap of $\mathcal{H}$ iff $\mathcal{S}$ is a normal subgroup of $(\mathcal{H}, \cdot_e)$ for every $e \in \mathcal{S}$.

Now we will discuss some properties of heaps related to heap morphism. Let ϕ be a morphism of heap from $\mathcal{H}$ to $\mathcal{H}'$. If $\mathcal{S}$ is a sub-heap of $\mathcal{H}$, then $\phi(\mathcal{S})$ is also a sub-heap of $\mathcal{H}'$. Furthermore, ϕ also preserves the normality of $\mathcal{S}$. Otherwise, if subset $\mathcal{T}$ of $\mathcal{H}'$ is a sub-heap, then subset $\phi^{-1}(\mathcal{T})$ of $\mathcal{H}$ is also sub-heap. Moreover, if $\mathcal{T}$ is normal in $\mathcal{H}'$, then $\phi^{-1}(\mathcal{T})$ is also normal in $\mathcal{H}$.

2.2 Graph theory

Our next discussion is the graph theory. We will present fundamental definitions of some terms in graph. A graph Γ contains a nonempty vertex set $V(\Gamma)$ and an edge set $E(\Gamma)$ which contains non-ordered pairs of distinct elements of $V(\Gamma)$. This edge set can be the empty set. Two distinct $a, b \in V(\Gamma)$ are called adjacent if there is an edge in $E(\Gamma)$ such that a and b are the endpoints. This single edge is denoted by (a, b) or (b, a). A path from a to b is a sequence of vertices $a = a_1 - a_2 - \cdots - a_n = b$ where a_i, a_{i+1} are adjacent and $a_i \neq a_j$ for $i \neq j$.

Furthermore, a cycle is a path in which the endpoints are the same. Let a, b be an arbitrary distinct vertices of Γ. Graph Γ is said to be connected if we can make a path from a to b in Γ. The distance $d(a, b)$ is the shortest path from a to b. The longest distance in Γ is called diameter of Γ, while the girth is the length of the shortest cycle. The degree of vertex a is the number of edge which is incident to a and it is denoted by $deg(a)$. The minimum and maximum degree of Γ are denoted by $\delta(\Gamma)$ and $\Delta(\Gamma)$ respectively. A decomposition of graph Γ is a family of edge-disjoint subgraphs $\Gamma_1, \Gamma_2, \cdots \Gamma_l$ such that $E(\Gamma) = \bigcup_{i=1}^{l} E(\Gamma_i)$ [4]. There is a topological index which is related to the degree of vertices of graph. It is called Zagreb index. There are two kinds on Zagreb indices, those are the first Zagreb index which is denoted by $M_1(\Gamma)$ and the second Zagreb index which is denoted by $M_2(\Gamma)$ [11]. Each formula of Zagreb indices are as follows,

$$M_1(\Gamma) = \sum_{a \in V(\Gamma)} (deg(a))^2, \tag{5}$$

$$M_2(\Gamma) = \sum_{(a,b) \in E(\Gamma)} deg(a) \cdot deg(b). \tag{6}$$

3 Results

We begin by defining a bipartite graph associated to elements and equivalence classes of a heap. Let $\mathcal{H}$ be a finite heap and $\mathcal{A} = \{\mathcal{S}|\mathcal{S}$ sub-heap of $\mathcal{H}\}$. A bipartite graph associated to elements and equivalence classes $\Gamma(\mathcal{H})$ is a graph with $\mathcal{H} \cup \mathcal{A}$ as the set of vertices and two distinct vertices $x \in \mathcal{H}$ and $\mathcal{S} \in \mathcal{A}$ are adjacent if $\overline{x}_{\mathcal{S}} = {}_{\mathcal{S}}\overline{x}$. We simply call graph $\Gamma(H)$ as a bipartite graph of $\mathcal{H}$ when no confusion arise.

Theorem 3.1. *Assume that $\mathcal{H}$ is a heap and $\Gamma(\mathcal{H})$ is the corresponding bipartite graph. Then, the following properties are satisfied;*

1. *Graph $\Gamma(\mathcal{H})$ is a connected graph.*
2. *Diameter of $\Gamma(\mathcal{H})$ is less than or equal to 4.*
3. *Girth of $\Gamma(\mathcal{H})$ is equal to 4.*

Proof.

1. Let A, B be any two vertices in $\Gamma(\mathcal{H})$. Then, there are some cases will be considered;

   Case 1: Let $A = x \in \mathcal{H}$ and $B = y \in \mathcal{H}$. Then we can make a path from x to y, that is $x - \{x\} - y$.

   Case 2: Let $A = x \in \mathcal{H}$ and $B = \mathcal{S}$ be the sub-heap of $\mathcal{H}$. We can make a path $x - \{x\} - s - \mathcal{S}$ for some $s \in \mathcal{S}$.

   Case 3: Let $A = \mathcal{S}_1$ and $B = \mathcal{S}_2$ be the sub-heaps of $\mathcal{H}$. The path between $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\mathcal{S}_1 - s_1 - \{s_1\} - s_2 - \mathcal{S}_2$ for some $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$.

2. **Case 1:** The maximum distance between any two vertices $x, y \in \mathcal{H}$ in $\Gamma(\mathcal{H})$ is equal to 2 since x is adjacent to sub-heap $\{x\}$ and $\{x\}$ is adjacent to y.

   Case 2: Assume that w is any element of $\mathcal{H}$ and $\mathcal{S}$ is an arbitrary element of $\mathcal{A}$. The maximum distance between these two elements is 3 since we have a path $w - \{w\} - z - \mathcal{S}$ for any $z \in \mathcal{S}$.

   Case 3: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be any elements of $\mathcal{A}$. The maximum distance between $\mathcal{S}_1$ and $\mathcal{S}_2$ is equal to 4. This is because we have a path $\mathcal{S}_1 - x - \{x\} - y - \mathcal{S}_2$ for any $x \in \mathcal{S}_1, y \in \mathcal{S}_2$.

3. The shortest cycle in $\Gamma(\mathcal{H})$ that we can make is $a - \{a\} - b - \{b\} - a$ for any $a, b \in \mathcal{H}$.

□

Theorem 3.2. *If $\mathcal{H}$ is a heap with $|\mathcal{H}| = p$ for some prime p, then $\Gamma(\mathcal{H}) = K_{p,p+1}$.*

Proof. If $|\mathcal{H}| = p$ for some prime p, then the order of arbitrary sub-heaps of $\mathcal{H}$ is 1 or p. Consequently, the sub-heaps of $\mathcal{H}$ are singleton or $\mathcal{H}$ itself. Thus, the family of sub-heaps of $\mathcal{H}$ is $\mathcal{A} = \{\{x_1\}, \{x_2\}, \dots, \{x_p\}, \mathcal{H}\}$ where $x_i \in \mathcal{H}$ for all $i = 1, 2, \dots, p$. Note that every sub-heaps of $\mathcal{H}$ is adjacent to every elements of $\mathcal{H}$. Hence, the result follows. □

Theorem 3.3. *Suppose that $\mathcal{H}$ is a heap with $|\mathcal{H}| = n$. Then, sub-heap $\mathcal{S}$ of $\mathcal{H}$ is normal iff $deg(\mathcal{S}) = n$.*

Proof. Let $\mathcal{S}$ be a normal sub-heap of $\mathcal{H}$. Then, $\overline{x}_{\mathcal{S}} = {}_{\mathcal{S}}\overline{x}$ for all $x \in \mathcal{H}$. Therefore, $deg(\mathcal{S}) = |\mathcal{H}|$. Conversely, assume that $deg(\mathcal{S}) = |\mathcal{H}|$. Then $\mathcal{S}$ is adjacent to x for all $x \in \mathcal{H}$ or equivalently $\overline{x}_{\mathcal{S}} = {}_{\mathcal{S}}\overline{x}$. □

Theorem 3.4. *Let ψ be a surjective morphism of heap from heap $\mathcal{H}$ to $\mathcal{H}'$ and $\mathcal{L}$ be a sub-heap of $\mathcal{H}$. If $deg(\mathcal{L}) = |\mathcal{H}|$ then $deg(\psi(\mathcal{L})) = |\mathcal{H}'|$.*

Proof. If $deg(\mathcal{L}) = |\mathcal{H}|$, then by Theorem 3.3, $\mathcal{L}$ is normal in $\mathcal{H}$. We thus get $\psi(\mathcal{L})$ is normal in $\mathcal{H}'$ by the surjectivity of ψ. □

Theorem 3.5. *Let $\mathcal{H}$ be a heap and $(\mathcal{H}, \cdot_e, e)$ be the group by fixing $e \in \mathcal{H}$ to be the identity. If $\Gamma(\mathcal{H})$ and $\Gamma'(\mathcal{H})$ are bipartite graphs of heap $\mathcal{H}$ and group $(\mathcal{H}, \cdot_e, e)$ respectively, then $\Gamma'(\mathcal{H})$ is a subgraph of $\Gamma(\mathcal{H})$.*

Proof. Note that, every subgroups of $\mathcal{H}$ is also sub-heaps of heap $\mathcal{H}$. It shows $V(\Gamma'(\mathcal{H})) \subseteq V(\Gamma(\mathcal{H}))$. Suppose that $(x, \mathcal{N}) \in E(\Gamma'(\mathcal{H}))$ where $x \in \mathcal{H}$ and $\mathcal{N}$ subgroup of $\mathcal{H}$, then $x\mathcal{N} = \mathcal{N}x$. This implies $\overline{x}_{\mathcal{N}} = {}_{\mathcal{N}}\overline{x}$ and hence $(x, \mathcal{N}) \in E(\Gamma(\mathcal{H}))$. □

Theorem 3.6. *Suppose that ψ is a heap morphism from $\mathcal{H}$ to $\mathcal{H}'$ and $\mathcal{T}$ be an arbitrary sub-heap of $\mathcal{H}'$. If ψ onto, then the following conditions satisfied;*

1. *If $\big(a, \psi^{-1}(\mathcal{T})\big) \in E\big(\Gamma(\mathcal{H})\big)$, then $\big(\psi(a), \mathcal{T}\big) \in E\big(\Gamma(\mathcal{H}')\big)$.*
2. *If $(b, \mathcal{T}) \in \Gamma(\mathcal{H}')$, then $\big(c, \psi^{-1}(\mathcal{T})\big) \in E\big(\Gamma(\mathcal{H})\big)$ for every $c \in \psi^{-1}(b)$.*

Proof.

1. We will prove that, $\overline{\psi(a)}_{\mathcal{T}} = {}_{\mathcal{T}}\overline{\psi(a)}$. Let x be any element of $\overline{\psi(a)}_{\mathcal{T}}$. Then there is $t \in \mathcal{T}$ which satisfies $[x, \psi(a), t] = t_1 \in \mathcal{T}$. By the surjectivity of ψ, we can write $\psi(z) = x$ and $\psi(s) = t$ for some $z \in \mathcal{H}, s \in \psi^{-1}(\mathcal{T})$. Then,
$$\begin{aligned} \big[x, \psi(a), t\big] &= t_1, \\ \big[\psi(z), \psi(a), \psi(s)\big] &= t_1, \\ \psi\big([z, a, s]\big) &= t_1. \end{aligned}$$
Hence, $[z, a, s] \in \psi^{-1}(\mathcal{T})$ which implies $z \in \overline{a}_{\psi^{-1}(\mathcal{T})} = {}_{\psi^{-1}(\mathcal{T})}\overline{a}$. Thus, $[s', a, z] \in \psi^{-1}(\mathcal{T})$ for some $s' \in \psi^{-1}(\mathcal{T})$. Note that,
$$\begin{aligned} [s', a, z] \in \psi^{-1}(\mathcal{T}) &\Longleftrightarrow \psi\big([s', a, z]\big) \in \mathcal{T} \\ &\Longleftrightarrow \big[\psi(s'), \psi(a), \psi(z)\big] \in \mathcal{T} \\ &\Longleftrightarrow \big[\psi(s'), \psi(a), x\big] \in \mathcal{T}, \end{aligned}$$
and $\psi(s') \in \mathcal{T}$ implying $x \in {}_{(\mathcal{T})}\overline{\psi(a)}$. Therefore $\overline{\psi(a)}_{\mathcal{T}} \subseteq {}_{(\mathcal{T})}\overline{\psi(a)}$. We can use similar way to prove ${}_{\mathcal{T}}\overline{\psi(a)} \subseteq \overline{\psi(a)}_{\mathcal{T}}$.
2. We begin by proving $\overline{c}_{\psi^{-1}(\mathcal{T})} = {}_{\psi^{-1}(\mathcal{T})}\overline{c}$. Take an element y of $\overline{c}_{\psi^{-1}(\mathcal{T})}$. Then, we have $[y, c, s] \in \psi^{-1}(\mathcal{T})$ for some $s \in \psi^{-1}(\mathcal{T})$ or equivalently $\big[\psi(y), b, \psi(s)\big] = \psi\big([y, c, s]\big) \in \mathcal{T}$ which means $\psi(y) \in \overline{b}_{\mathcal{T}} = {}_{\mathcal{T}}\overline{b}$. Consequently, there exists $t' \in \mathcal{T}$ such that $\big[t', b, \psi(y)\big] \in \mathcal{T}$. Since ψ is surjective, we have
$$\begin{aligned} \big[t', b, \psi(y)\big] &= \big[\psi(s'), \psi(d), \psi(y)\big], \ \exists s' \in \psi^{-1}(t') \subseteq \psi^{-1}(\mathcal{T}), d \in \psi^{-1}(b), \\ &= \psi\big([s', d, y]\big) \in \mathcal{T}, \end{aligned}$$

which means $[s', d, y] \in \psi^{-1}(\mathcal{T})$. Note that, $c, d \in \psi^{-1}(b)$ where $\psi^{-1}(b) = \ker_b(\psi)$ is normal in $\mathcal{H}$. Thus, ${}_{\psi^{-1}(b)}\overline{c} = \overline{c}_{\psi^{-1}(b)} = \overline{d}_{\psi^{-1}(b)} = {}_{\psi^{-1}(b)}\overline{d}$. Then, we can write $[z, c, d] = g$ for some $z, g \in \psi^{-1}(b)$ or equivalently $d = [c, z, g]$. By subtituing $d = [c, z, g]$ to $[s', d, y]$, we get

$$\begin{aligned}\big[s', [c, z, g], y\big] &= \big[s', g, [z, c, y]\big], \\ &= \big[[s', g, z], c, y\big] \in \psi^{-1}(\mathcal{T}).\end{aligned}$$

Note that, $\psi\big([s', g, z]\big) = \big[\psi(s'), \psi(g), \psi(z)\big] = \big[t', b, b\big] = t' \in \mathcal{T}$. We can conclude that, $\big[s', g, z\big] \in \psi^{-1}(\mathcal{T})$. Therefore, we have $y \in {}_{\psi^{-1}(\mathcal{T})}\overline{c}$. We can use analogous way to prove ${}_{\psi^{-1}(\mathcal{T})}\overline{c} \subseteq \overline{c}_{\psi^{-1}(\mathcal{T})}$.

□

Corollary 3.1. *Suppose that ψ is a heap morphism from $\mathcal{H}$ to $\mathcal{H}'$ and*

$$b_1 - \mathcal{T}_1 - b_2 - \mathcal{T}_2 - \cdots - b_n - \mathcal{T}_n, \tag{7}$$

is a path in $\Gamma(\mathcal{H}')$. If ψ is onto, then there exists a path in $\Gamma(\mathcal{H})$ which corresponds to path (7).

Proof. Assume that, we have path (7). Since b_i is adjacent to $\mathcal{T}_i$ and b_j is adjacent to $\mathcal{T}_j$ for $i \neq j$, by Theorem 3.6, we have a_i is adjacent to $\psi^{-1}(\mathcal{T}_i)$ and a_j is adjacent to $\psi^{-1}(\mathcal{T}_j)$ for some $a_i \in \psi^{-1}(b_i)$, $a_j \in \psi^{-1}(b_j)$. If $a_i = a_j$ and $\psi^{-1}(\mathcal{T}_i) = \psi^{-1}(\mathcal{T}_j)$, then $b_i = \psi(a_i) = \psi(a_j) = b_j$ and $\mathcal{T}_i = \psi(\psi^{-1}(\mathcal{T}_i)) = \psi(\psi^{-1}(\mathcal{T}_j)) = \mathcal{T}_j$ which is impossible. Hence, we have a path

$$a_1 - \psi^{-1}(\mathcal{T}_1) - a_2 - \psi^{-1}(\mathcal{T}_2) - \cdots - a_n - \psi^{-1}(\mathcal{T}_n), \tag{8}$$

in $\Gamma(\mathcal{H})$. □

Corollary 3.2. *Suppose that ψ is a heap morphism from $\mathcal{H}$ to $\mathcal{H}'$ and*

$$b_1 - \mathcal{T}_1 - b_2 - \mathcal{T}_2 - \cdots - b_n - \mathcal{T}_n - b_1, \tag{9}$$

is a cycle of $\Gamma(\mathcal{H}')$. If ψ is onto then,

$$a_1 - \psi^{-1}(\mathcal{T}_1) - a_2 - \psi^{-1}(\mathcal{T}_2) - \cdots - a_n - \psi^{-1}(\mathcal{T}_n) - a_1, \tag{10}$$

is a cycle of $\Gamma(\mathcal{H})$ for some $a_i \in \psi^{-1}(b_i)$.

Proof. The proof is obvious. □

Theorem 3.7. *If ϕ is a surjective morphism from heap $\mathcal{H}$ to $\mathcal{H}'$, then the following conditions hold;*

1. *If x is any element of $\mathcal{H}$, then $deg(x) \geq deg(\phi(x))$.*
2. *If $\mathcal{T}$ is any sub-heaps of $\mathcal{H}'$, then $deg(\phi^{-1}(\mathcal{T})) \geq deg(\mathcal{T})$.*

Proof.

1. Let $deg(\phi(x)) = m$ and $\phi(x)$ be adjacent to $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$ (see Figure 1). Then, by Theorem 3.6, x is adjacent to $\phi^{-1}(\mathcal{T}_1), \phi^{-1}(\mathcal{T}_2), \ldots, \phi^{-1}(\mathcal{T}_m)$. Suppose that, $\phi^{-1}(\mathcal{T}_i) = \phi^{-1}(\mathcal{T}_j)$ for $i \neq j$. Then, there exists two distinct edges, $\big(x, \phi^{-1}(\mathcal{T}_i)\big)$ and $\big(x, \phi^{-1}(\mathcal{T}_j)\big)$ in $\Gamma(\mathcal{H})$. Hence, $\phi^{-1}(\mathcal{T}_i) \neq \phi^{-1}(\mathcal{T}_j)$. Note that, it is not guaranteed that if $(x, \mathcal{S}) \in E(\Gamma(\mathcal{H}))$ then, $\big(\phi(x), \phi(\mathcal{S})\big) \in E\big(\Gamma(\mathcal{H}')\big)$. Therefore, $deg(x) \geq deg(\phi(x))$.

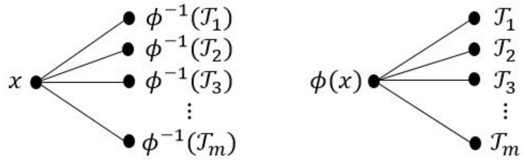

Figure 1: Vertex x and $\phi(x)$ of $\Gamma(\mathcal{H})$ and $\Gamma(\mathcal{H}')$ respectively.

2. Let, $deg(\mathcal{T}) = m$ and $\mathcal{T}$ be adjacent to $y_1, y_2, y_3, \dots, y_m$. Then, by Theorem 3.6, for every $x_i \in \phi^{-1}(y)$, x_i is adjacent to $\phi^{-1}(\mathcal{T})$. It might be $|\phi^{-1}(y)| \geq 1$. Note that, if $x_i = x_j$, then $y_i = \phi(x_i) = \phi(x_j) = y_j$ which is impossible. Therefore, we can conclude that $deg(\phi^{-1}(\mathcal{T})) \geq deg(\mathcal{T})$.

□

Theorem 3.8. *Suppose that ψ is a heap morphism from $\mathcal{H}$ to $\mathcal{H}'$ which is onto. Then,*

1. *if $x_1, x_2 \in \mathcal{H}$ and $x_1 \neq x_2$, then $d(x_1, x_2) \geq d\big(\psi(x_1), \psi(x_2)\big)$.*
2. *if $y \in \mathcal{H}'$ and $\mathcal{T}$ is a sub-heap of $\mathcal{H}'$, then $d\big(x, \psi^{-1}(\mathcal{T})\big) = d(y, \mathcal{T})$ for every $x \in \psi^{-1}(y)$.*
3. *if $\mathcal{T}_1, \mathcal{T}_2$ are sub-heaps of $\mathcal{H}'$, then $d\big(\psi^{-1}(\mathcal{T}_1), \psi^{-1}(\mathcal{T}_2)\big) = d(\mathcal{T}_1, \mathcal{T}_2)$.*

Proof.

1. Note that, $d(x_1, x_2) = 2$ with the shortest path $x_1 - \mathcal{H} - x_2$. It is possible that $\psi(x_1) = \psi(x_2)$. In this case, we have $d\big(\psi(x_1), \psi(x_2)\big) = 0$. If $\psi(x_1) \neq \psi(x_2)$, then $d\big(\psi(x_1), \psi(x_2)\big) = 2$ with the path $\psi(x_1) - \mathcal{H}' - \psi(x_2)$. Therefore, the result follows.
2. Let $y \in \mathcal{H}'$ and $\mathcal{T}$ be a sub-heap of $\mathcal{H}'$;
   (a) If y is adjacent to $\mathcal{T}$, then $d(y, \mathcal{T}) = 1$. By Theorem 3.6, x is adjacent to $\psi^{-1}(\mathcal{T})$ for every $x \in \psi^{-1}(y)$. Hence, $d\big(x, \psi^{-1}(\mathcal{T})\big) = 1$.
   (b) If y is not adjacent to $\mathcal{T}$, then $d(y, \mathcal{T}) = 3$ with the shortest path $\mathcal{T} - t - \{t\} - y$ for some $t \in \mathcal{T}$. Suppose that there exists $x \in \psi^{-1}(y)$ such that x is adjacent to $\phi^{-1}(\mathcal{T})$. By Theorem 3.6, $\psi(x) = y$ is adjacent to $\mathcal{T}$ which is a contradiction. Thus, for every $x \in \psi^{-1}(y)$, x is not adjacent to $\mathcal{T}$. Therefore, the shortest path between x and $\psi^{-1}(\mathcal{T})$ is $x - \{s\} - s - \psi^{-1}(\mathcal{T})$ where $\psi(s) = t$ and it implies that $d\big(x, \psi^{-1}(\mathcal{T})\big) = 3$.
3. There are some cases which will be considered.
   (a) Let $y \in \mathcal{T}_1 \cap \mathcal{T}_2$. Then, $\mathcal{T}_1 - y - \mathcal{T}_2$ is the shortest path between $\mathcal{T}_1$ and $\mathcal{T}_2$. Hence, $d(\mathcal{T}_1, \mathcal{T}_2) = 2$. Note that, by Theorem 3.6, x is adjacent to $\psi^{-1}(\mathcal{T}_1)$ and $\psi^{-1}(\mathcal{T}_2)$ for every $x \in \psi^{-1}(y)$. Hence $d\big(\psi^{-1}(\mathcal{T}_1), \psi^{-1}(\mathcal{T}_2)\big) = 2$.
   (b) Let $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ and b be adjacent to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then, $d(\mathcal{T}_1, \mathcal{T}_2) = 2$. According to Theorem 3.6, a is adjacent to $\psi^{-1}(\mathcal{T}_1)$ and $\psi^{-1}(\mathcal{T}_2)$ for every $a \in \psi^{-1}(b)$. Therefore, $d\big(\psi^{-1}(\mathcal{T}_1), \psi^{-1}(\mathcal{T}_2)\big) = 2$.
   (c) Assume that $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ and there is no $b \in \mathcal{H}'$ such that b is adjacent to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then, $d(\mathcal{T}_1, \mathcal{T}_2) = 4$ with the shortest path $\mathcal{T}_1 - t_1 - \{t_1\} - t_2 - \mathcal{T}_2$ for any $t_1 \in \mathcal{T}_1$ and $t_2 \in \mathcal{T}_2$. Suppose that $x \in \psi^{-1}(\mathcal{T}_1) \cap \psi^{-1}(\mathcal{T}_2)$. We have $\phi(x) \in \mathcal{T}_1 \cap \mathcal{T}_2$ which is a contradiction. This makes $\psi^{-1}(\mathcal{T}_1) \cap \psi^{-1}(\mathcal{T}_2) = \emptyset$. Now, suppose that $x \in \mathcal{H}$ is adjacent to $\psi^{-1}(\mathcal{T}_1)$ and $\psi^{-1}(\mathcal{T}_2)$. According to Theorem 3.6, it implies $\psi(x)$ is adjacent to $\mathcal{T}_1$ and $\mathcal{T}_2$. This

is also a contradiction. Therefore, the shortest path between $\psi^{-1}(\mathcal{T}_1)$ and $\psi^{-1}(\mathcal{T}_2)$ is $\psi^{-1}(\mathcal{T}_1) - s_1 - \{s_1\} - s_2 - \psi^{-1}(\mathcal{T}_2)$ and thus $d\big(\psi^{-1}(\mathcal{T}_1), \psi^{-1}(\mathcal{T}_2)\big) = 4$.

□

Theorem 3.9. *Suppose that ψ is a morphism of heaps from $\mathcal{H}$ to $\mathcal{H}'$. If ψ onto, then the following properties are hold;*

1. *If $\delta(\Gamma(\mathcal{H})) = deg(x)$ and $\delta(\Gamma(\mathcal{H}')) = deg(\phi(w))$, then $\delta(\Gamma(\mathcal{H})) \leq \delta(\Gamma(\mathcal{H}'))$.*
2. *If $\delta(\Gamma(\mathcal{H})) = deg(x)$ and $\delta(\Gamma(\mathcal{H}')) = deg(\mathcal{T})$, then $\delta(\Gamma(\mathcal{H})) \leq \delta(\Gamma(\mathcal{H}'))$.*

Proof.

1. $\delta(\Gamma(\mathcal{H})) = deg(x)$ and $\delta(\Gamma(\mathcal{H}')) = deg(\phi(w))$ for some $w \in \mathcal{H}$. According to Theorem 3.7, we have $deg(\phi(x)) \leq deg(x)$. Since $deg(\phi(w)) = \delta(\Gamma(\mathcal{H}'))$, then $deg(\phi(w)) \leq deg(\phi(x))$. Therefore, we have $deg(\phi(w)) \leq deg(x)$ which implies $\delta(\Gamma(\mathcal{H}')) \leq \delta(\Gamma(\mathcal{H}))$.
2. Let $\delta(\Gamma(\mathcal{H})) = deg(x)$ for some $x \in \mathcal{H}$ and $\delta(\Gamma(\mathcal{H}')) = deg(\mathcal{T})$ for some sub-heap $\mathcal{T}$ of $\mathcal{H}'$. It implies $deg(\mathcal{T}) \leq deg(\phi(x))$. According to Theorem 3.7, we have $deg(\phi(x)) \leq deg(x)$. Therefore, we obtain the inequality $\delta(\Gamma(\mathcal{H}')) = deg(\mathcal{T}) \leq deg(x) = \delta(\Gamma(\mathcal{H}))$.

□

Theorem 3.10. *Let $\phi : \mathcal{H} \longrightarrow \mathcal{H}'$ be a morphism of heaps which is onto. Then $\Delta(\Gamma(\mathcal{H})) \geq \Delta(\Gamma(\mathcal{H}'))$.*

Proof.

Case 1: Let $\Delta(\Gamma(\mathcal{H})) = deg(x)$ and $\Delta(\Gamma(\mathcal{H}')) = deg(\phi(y))$ for some $x, y \in \mathcal{H}$. On the contrary, assume that $n = deg(x) < deg(\phi(y)) = m$. Then, $\phi(y)$ is adjacent to $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$, where $\mathcal{T}_i$ are sub-heaps of $\mathcal{H}'$. According to Theorem 3.6, $y \in \phi^{-1}(\phi(y))$ is adjacent to $\phi^{-1}(\mathcal{T}_i)$ for all $i = 1, 2, \ldots, m$. If $\phi^{-1}(\mathcal{T}_i) = \phi^{-1}(\mathcal{T}_j)$, then $\mathcal{T}_i = \mathcal{T}_j$ which is impossible. Thus, $\phi^{-1}(\mathcal{T}_i) \neq \phi^{-1}(\mathcal{T}_j)$ for $i \neq j$. Note that, y might be adjacent to sub-heap $\mathcal{S}$ of $\mathcal{H}$, where $\mathcal{S} \neq \mathcal{T}_i$ for every $i = 1, 2, \ldots, m$. Therefore, we obtain $deg(y) \geq m > n = deg(x)$ which contradicts to $deg(x)$ is maximum in $\Gamma(\mathcal{H})$.

Case 2: Let $\Delta(\Gamma(\mathcal{H})) = deg(x)$ for some $x \in \mathcal{H}$ and $\Delta(\Gamma(\mathcal{H}')) = deg(\mathcal{T})$ for some sub-heap $\mathcal{T}$ of $\mathcal{H}'$. Suppose that, $n = deg(x) < deg(\mathcal{T}) = m$. Assume that, $\mathcal{T}$ is adjacent to $w_1, w_2, \ldots, w_m \in \mathcal{H}'$. By Theorem 3.6, for every i, there exists $a_i \in \phi^{-1}(w_i)$ such that a_i is adjacent to $\phi^{-1}(\mathcal{T})$. If $a_i = a_j$, then $w_i = \phi(a_i) = \phi(a_j) = w_j$ for $i \neq j$ which is impossible. If $\phi^{-1}(\mathcal{T})$ is adjacent to $c \in \mathcal{H}$ where $c \neq a_i$ for all i, then by Theorem 3.6, we obtain $\phi(\phi^{-1}(\mathcal{T})) = \mathcal{T}$ is adjacent to $\phi(c)$. Since $deg(\mathcal{T}) = m$, then $\phi(c) = w_j$ for some j. Hence, we have $deg(\phi^{-1}(\mathcal{T})) \geq m > n = deg(x)$. This is a contradiction since $deg(x)$ is maximum.

Case 3: Let $\Delta(\Gamma(\mathcal{H})) = deg(\mathcal{S})$ for some sub-heap $\mathcal{S}$ of $\mathcal{H}$ and $\Delta(\Gamma(\mathcal{H}')) = deg(z)$ for some $z \in \mathcal{H}'$. Suppose that, $n = deg(\mathcal{S}) < deg(z) = m$. Let z be adjacent to sub-heaps $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$ of $\mathcal{H}'$. By Theorem 3.6, for any $y \in \phi^{-1}(z)$, y is adjacent to $\phi^{-1}(\mathcal{T}_1), \phi^{-1}(\mathcal{T}_2), \ldots, \phi^{-1}(\mathcal{T}_m)$. It is clear that $\phi^{-1}(\mathcal{T}_i) \neq \phi^{-1}(\mathcal{T}_j)$ for $i \neq j$. Note that, y might be adjacent to some sub-heap $\mathcal{S}$ of $\mathcal{H}$ where $\mathcal{S} \neq \phi^{-1}(\mathcal{T}_i)$ for all i. Hence, we get $deg(y) \geq m > n = deg(\mathcal{S})$. This is a contradiction.

Case 4: Let $\Delta(\Gamma(\mathcal{H})) = deg(\mathcal{S})$ and $\Delta(\Gamma(\mathcal{H}')) = deg(\mathcal{T})$ for some sub-heaps $\mathcal{S}$ of $\mathcal{H}$ and sub-heap $\mathcal{T}$ of $\mathcal{H}'$. Assume that $n = deg(\mathcal{S}) < deg(\mathcal{T}) = m$. Let $\mathcal{T}$ be adjacent to $z_1, z_2, \cdots, z_m \in \mathcal{H}'$. According Theorem 3.6, for every $i = 1, 2, \cdots, m$, there exists $x_i \in \phi^{-1}(z_i)$ such that x_i is adjacent to $\phi^{-1}(\mathcal{T})$. If $x_i = x_j$, then $z_i = \phi(x_i) = \phi(x_j) = z_j$ which is impossible. Thus $x_i \neq x_j$ for $i \neq j$. Note that if $\phi^{-1}(\mathcal{T})$ is adjacent to w for $w \neq x_i$, then by Theorem 3.6, $\phi(\phi^{-1}(\mathcal{T})) = \mathcal{T}$ is adjacent to $\phi(w)$. Since $deg(\mathcal{T}) = m$, then $\phi(w) = z_j$ for some j. Hence we have $deg(\phi^{-1}(\mathcal{T})) \geq m > n = deg(\mathcal{S})$ which is a contradiction.

Therefore $\Delta(\Gamma(\mathcal{H})) \geq \Delta(\Gamma(\mathcal{H}'))$. □

Theorem 3.11. *Let $\phi : \mathcal{H} \longrightarrow \mathcal{H}'$ be a morphism of heaps. If ϕ is onto and $\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_r$ is a decomposition of a graph $\Gamma(\mathcal{H}')$, then $\Gamma_1, \Gamma_2, \ldots, \Gamma_r, \Lambda$ is a decomposition of a graph $\Gamma(\mathcal{H})$ where*

$$E(\Gamma_i) = \Big\{ \Big(a, \phi^{-1}(\mathcal{T})\Big) \Big| \Big(\phi(a), \mathcal{T}\Big) \in E(\Gamma'_i) \Big\}, \tag{11}$$

and

$$E(\Lambda) = \Big\{ \Big(x, \mathcal{S}\Big) \Big| \mathcal{S} \neq \phi^{-1}(\mathcal{T}), \forall \mathcal{T} \text{ sub-heap of } \mathcal{H}' \Big\}. \tag{12}$$

Proof. First, we will prove that $E(\Gamma_i) \subseteq E(\Gamma(\mathcal{H}))$ for every $i = 1, 2, \ldots, r$. Let, $\big(a, \phi^{-1}(\mathcal{T})\big)$ be an arbitrary element of $E(\Gamma_i)$. Then, by (11), $(\phi(a), \mathcal{T}) \in E(\Gamma'_i)$. By Theorem 3.6, $\big(a, \phi^{-1}(\mathcal{T})\big) \in E\big(\Gamma(\mathcal{H})\big)$. Hence, $E(\Gamma_i)$ is a subgraph of $\Gamma(\mathcal{H})$ for every $i = 1, 2, 3, \ldots, r$. Note that, from (12), Λ is obviously a subgraph of $\Gamma(\mathcal{H})$. Next, we will prove that $E(\Gamma_i) \cap E(\Gamma_j) = \emptyset$ for $i \neq j$ and $E(\Gamma_i) \neq \Lambda$ for every $i = 1, 2, \ldots, r$. Suppose that, $\big(a, \phi^{-1}(\mathcal{T})\big) \in E(\Gamma_i) \cap E(\Gamma_j)$. Then, $\big(\phi(a), \mathcal{T}\big) \in E(\Gamma'_i) \cap E(\Gamma'_j)$ which is a contradiction since $\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_r$ is a decomposition of a graph $\Gamma(\mathcal{H}')$. Now, suppose that $(x, \mathcal{S}) \in E(\Gamma_i) \cap E(\Lambda)$ for some i.

Then, by (11) and (12), $(x, \mathcal{S}) = \big(a, \phi^{-1}(\mathcal{T})\big)$ for some sub-heap $\mathcal{T}$ of $\mathcal{H}'$ and $\mathcal{S} \neq \phi^{-1}(\mathcal{T})$ for every sub-heap $\mathcal{T}$ of $\mathcal{H}'$. This is impossible. Therefore, $\Gamma_1, \Gamma_2, \ldots, \Gamma_r, \Lambda$ are edge-disjoint subgraphs of $\Gamma(\mathcal{H})$. Take any edge $(b, \mathcal{U}) \in E(\Gamma(\mathcal{H}))$. If $\mathcal{U} = \phi^{-1}(\mathcal{T})$ for some sub-heap $\mathcal{T}$ of $\mathcal{H}'$, then by Theorem 3.6, $\big(\phi(b), \phi(\mathcal{U})\big) = \big(\phi(b), \mathcal{T}\big) \in E(\Gamma(\mathcal{H}'))$. Since $\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_r$ is a decomposition of a graph $\Gamma(\mathcal{H}')$, then $\big(\phi(b), \mathcal{T}\big) \in E(\Gamma'_i)$ for some i. Hence, $\big(b, \phi^{-1}(\mathcal{T})\big) = (b, \mathcal{U}) \in E(\Gamma_i)$. Now, if $\mathcal{U} \neq \phi^{-1}(\mathcal{T})$ for all sub-heap $\mathcal{T}$ of $\mathcal{H}'$, then $(b, \mathcal{U}) \in E(\Lambda)$. Therefore, we can conclude that $\Gamma_1, \Gamma_2, \ldots, \Gamma_r, \Lambda$ is a decomposition of a graph $\Gamma(\mathcal{H})$. □

Lemma 3.1. *Let $\phi : \mathcal{H} \longrightarrow \mathcal{H}'$ be a morphism of heaps which is onto. Then, $|E(\Gamma(\mathcal{H}'))| \leq |E(\Gamma(\mathcal{H}))|$.*

Proof. Let $(y, \mathcal{T}) \in E\big(\Gamma(\mathcal{H}')\big)$. Based on Theorem 3.6, $\big(x, \phi^{-1}(\mathcal{T})\big) \in E\big(\Gamma(\mathcal{H})\big)$ for every $x \in \phi^{-1}(y)$. It is possible that $|\phi^{-1}(y)| \geq 1$. It shows that the pre-image of $(y, \mathcal{T})$ in $E(\Gamma(\mathcal{H}))$ can be more than one edge. Now assume that $(x, \mathcal{S})$ is any edge in $\Gamma(\mathcal{H})$. If $\mathcal{S} \neq \phi^{-1}(\mathcal{T})$ for every sub-heap $\mathcal{T}$ of $\mathcal{H}'$, then it is not guaranteed that $\phi(x)$ is adjacent to $\phi(\mathcal{S})$. Therefore, we can conclude that $|E(\Gamma(\mathcal{H}'))| \leq |E(\Gamma(\mathcal{H}))|$. □

Theorem 3.12. *Let $\phi : \mathcal{H} \longrightarrow \mathcal{H}'$ be a morphism of heaps which is onto. Then the following conditions are satisfied;*

1. $M_1(\Gamma(\mathcal{H}')) \leq M_1(\Gamma(\mathcal{H}))$.
2. $M_2(\Gamma(\mathcal{H}')) \leq M_2(\Gamma(\mathcal{H}))$.

Proof.

1. Note that since ϕ is onto, $y = \phi(x)$ where $x \in \mathcal{H}$. Based on Theorem 3.7, we have the following inequality,

$$\begin{aligned} M_1(\Gamma(\mathcal{H}')) &= \sum_{y \in \mathcal{H}'} \Big(deg(y)\Big)^2 + \sum_{\mathcal{T} \text{ sub-heap of } \mathcal{H}'} \Big(deg(\mathcal{T})\Big)^2 \\ &\leq \sum_{x \in \mathcal{H}} \Big(deg(x)\Big)^2 + \sum_{\mathcal{T} \text{ sub-heap of } \mathcal{H}'} \Big(deg(\phi^{-1}(\mathcal{T}))\Big)^2 \\ &\leq \sum_{x \in \mathcal{H}} \Big(deg(x)\Big)^2 + \sum_{\mathcal{T} \text{ sub-heap of } \mathcal{H}'} \Big(deg(\phi^{-1}(\mathcal{T}))\Big)^2 + \sum_{\mathcal{S} \neq \phi^{-1}(\mathcal{T})} \Big(deg(\mathcal{S})\Big)^2 \\ &= M_1(\Gamma(\mathcal{H})). \end{aligned} \tag{13}$$

2. Note that by Theorem 3.6, for every edge $\big(\phi(a), \mathcal{T}\big) \in E\big(\Gamma(\mathcal{H}')\big)$, there exists some edges $\big(a, \phi^{-1}(\mathcal{T})\big) \in E(\Gamma(\mathcal{H}))$. It implies that,

$$\begin{aligned} M_2(\Gamma(\mathcal{H}')) &= \sum_{(\phi(a),\mathcal{T}) \in E(\Gamma(\mathcal{H}'))} deg\big(\phi(a)\big) \cdot deg(\mathcal{T}) \\ &\leq \sum_{(\phi(a),\mathcal{T}) \in E(\Gamma(\mathcal{H}'))} deg(a) \cdot deg\big(\phi^{-1}(\mathcal{T})\big) \\ &\leq \sum_{(a,\phi^{-1}(\mathcal{T})) \in E(\Gamma(\mathcal{H}))} |\phi^{-1}\big(\phi(a)\big)| \cdot deg(a) \cdot deg\big(\phi^{-1}(\mathcal{T})\big) \\ &\leq \sum_{(a,\phi^{-1}(\mathcal{T})) \in E(\Gamma(\mathcal{H}))} |\phi^{-1}\big(\phi(a)\big)| \cdot deg(a) \cdot deg\big(\phi^{-1}(\mathcal{T})\big) \\ &\quad + \sum_{\substack{(x,(\mathcal{S})) \in E(\Gamma(\mathcal{H})) \\ \mathcal{S} \neq \phi^{-1}(\mathcal{T})}} deg(x) \cdot deg(\mathcal{S}) \\ &= M_2\big(\Gamma(\mathcal{H})\big). \end{aligned} \tag{14}$$

□

Let $(\mathcal{H}_1, [-,-,-]_1)$ and $(\mathcal{H}_2, [-,-,-]_2)$ be heaps. We can construct a new heap $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ with the following ternary operation

$$\begin{aligned} &[-,-,-] : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H} \\ &\qquad \big[(a_1,a_2),(b_1,b_2),(c_1,c_3)\big] \mapsto \big[[a_1,b_1,c_1]_1,[a_2,b_2,c_2]_2\big]. \end{aligned} \tag{15}$$

Every sub-heap of $\mathcal{H}$ can be written as $\mathcal{S}_1 \times \mathcal{S}_2$ where $\mathcal{S}_1$ and $\mathcal{S}_2$ are sub-heaps of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Furthermore, every normal sub-heaps of $\mathcal{H}$ can be written as $\mathcal{N}_1 \times \mathcal{N}_2$ for $\mathcal{N}_1$ and $\mathcal{N}_2$ are normal sub-heaps of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. In the next properties, we will discuss the association between graph $\Gamma(\mathcal{H})$, $\Gamma(\mathcal{H}_1)$, and $\Gamma(\mathcal{H}_2)$.

Theorem 3.13. *Let $\mathcal{H}_1, \mathcal{H}_2$ be heaps and $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. An element $(x_1,x_2) \in \mathcal{H}$ is adjacent to $\mathcal{S}_1 \times \mathcal{S}_2$ in $\Gamma(\mathcal{H})$ iff x_1 is adjacent to $\mathcal{S}_1$ in $\Gamma(\mathcal{H}_1)$ and x_2 is adjacent to $\mathcal{S}_2$ in $\Gamma(\mathcal{H}_2)$.*

Proof. Assume that, $(x_1,x_2) \in \mathcal{H}$ is adjacent to $\mathcal{S}_1 \times \mathcal{S}_2$. This means that $\overline{(x_1,x_2)}_{\mathcal{S}_1 \times \mathcal{S}_2} = {}_{\mathcal{S}_1 \times \mathcal{S}_2}\overline{(x_1,x_2)}$. Now, let $a_1 \in \overline{x_1}_{\mathcal{S}_1}$. Then, $[x_1,a_1,s_1]_1 \in \mathcal{S}_1$ for some $s_1 \in \mathcal{S}_1$. We can always write $t \in \mathcal{S}_2$ as

$[x_2, x_2, t]_2$. Consequently, we have

$$\big[(x_1, x_2), (a_1, x_2), (s_1, t)\big] = \big([x_1, a_1, s_1]_1, [x_2, x_2, t]_2\big) \in \mathcal{S}_1 \times \mathcal{S}_2, \tag{16}$$

which means $(a_1, x_2) \in \overline{(x_1, x_2)}_{\mathcal{S}_1 \times \mathcal{S}_2} = {}_{\mathcal{S}_1 \times \mathcal{S}_2}\overline{(x_1, x_2)}$. Thus, there exists $(t_1, t_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ such that,

$$\begin{aligned} \underbrace{[(t_1, t_2), (a_1, x_2), (x_1, x_2)]}_{\in \mathcal{S}_1 \times \mathcal{S}_2} &= ([t_1, a_1, x_1]_1, [t_2, x_2, x_2]_2), \\ &= ([t_1, a_1, x_1], t_2). \end{aligned} \tag{17}$$

Therefore, we can conclude that $[t_1, a_1, x_1] \in \mathcal{S}_1$ or equivalently $a_1 \in {}_{\mathcal{S}_1}\overline{x_1}$.

Now, take any element $b_1 \in {}_{\mathcal{S}_1}\overline{x_1}$. Then, there exists $s' \in \mathcal{S}_1$ such that $[s', x_1, b_1]_1 \in \mathcal{S}_1$. Note that, for every $t' \in \mathcal{S}_2$ can be written as $t' = [t', x_2, x_2]_2$. Thus, we have

$$\big[(s', t'), (x_1, x_2), (b_1, x_2)\big] = \big([s', x_1, b_1]_1, [t', x_2, x_2]_2\big) \in \mathcal{S}_1 \times \mathcal{S}_2, \tag{18}$$

which means $(b_1, x_2) \in {}_{\mathcal{S}_1 \times \mathcal{S}_2}\overline{(x_1, x_2)} = \overline{(x_1, x_2)}_{\mathcal{S}_1 \times \mathcal{S}_2}$. Consequently, there exists $(u_1, u_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ such that,

$$\begin{aligned} \underbrace{[(x_1, x_2), (b_1, x_2), (u_1, u_2)]}_{\in \mathcal{S}_1 \times \mathcal{S}_2} &= ([x_1, b_1, u_1]_1, [x_2, x_2, u_2]_2), \\ &= ([x_1, b_1, u_1]_1, u_2). \end{aligned} \tag{19}$$

This implies $[x_1, b_1, u_1] \in \mathcal{S}_1$ for some $u_1 \in \mathcal{S}_1$ or equivalently $b_1 \in \overline{x_1}_{\mathcal{S}_1}$. We have thus proved that x_1 is adjacent to $\mathcal{S}_1$. By using similar way, we can prove that x_2 is adjacent to $\mathcal{S}_2$.

Assume that, x_1 is adjacent to $\mathcal{S}_1$ and x_2 is adjacent to $\mathcal{S}_2$. Our next objective is to prove that $(x_1, x_2) \in \mathcal{H}$ is adjacent to $\mathcal{S}_1 \times \mathcal{S}_2$. Assume that, $(c_1, c_2) \in \overline{(x_1, x_2)}_{\mathcal{S}_1 \times \mathcal{S}_2}$. Then, $\big[(c_1, c_2), (x_1, x_2), (v_1, v_2)\big] \in \mathcal{S}_1 \times \mathcal{S}_2$ for some $(v_1, v_2) \in \mathcal{S}_1 \times \mathcal{S}_2$. The following equality,

$$\big[(c_1, c_2), (x_1, x_2), (v_1, v_2)\big] = \big([c_1, x_1, v_1]_1, [c_2, x_2, v_2]_2\big), \tag{20}$$

shows that $[c_1, x_1, v_1]_1 \in \mathcal{S}_1$ and $[c_2, x_2, v_2]_2 \in \mathcal{S}_2$. It implies that $c_1 \in \overline{x_1}_{\mathcal{S}_1}$ and $c_2 \in \overline{x_2}_{\mathcal{S}_2}$. By the adjacency of x_1 to $\mathcal{S}_1$ and x_2 to $\mathcal{S}_2$, we have $c_1 \in {}_{\mathcal{S}_1}\overline{x_1}$ and $c_2 \in {}_{\mathcal{S}_2}\overline{x_2}$. Then, we can write $[w_1, x_1, c_1] \in \mathcal{S}_1$ and $[w_2, x_2, c_2] \in \mathcal{S}_2$ for some $w_1 \in \mathcal{S}_1, w_2 \in \mathcal{S}_2$ and consequently,

$$\underbrace{\big([w_1, x_1, c_1]_1, [w_2, x_2, c_2]_2\big)}_{\in \mathcal{S}_1 \times \mathcal{S}_2} = \big[(w_1, w_2), (x_1, x_2), (c_1, c_2)\big]. \tag{21}$$

This means that $(c_1, c_2) \in {}_{\mathcal{S}_1 \times \mathcal{S}_2}\overline{(x_1, x_2)}$. We can use analogous way to prove that,

$$_{\mathcal{S}_1 \times \mathcal{S}_2}\overline{(x_1, x_2)} \subseteq \overline{(x_1, x_2)}_{\mathcal{S}_1 \times \mathcal{S}_2}.$$

□

By Theorem 3.13, it is obvious that $\Gamma(\mathcal{H}_1 \times \mathcal{H}_2)$ is equal to the tensor product of graph $\Gamma(\mathcal{H}_1)$ and $\Gamma(\mathcal{H}_2)$.

4 Conclusions

Bipartite graph associated to elements and cosets of groups have been developed to heaps. The bipartite graph associated to elements and equivalence classes of heaps is a connected graph with diameter less than or equal to 4 and girth equal to 4. We also identify the relation between bipartite graph of heaps and groups. We obtain that the bipartite graph of groups is a subgraph of the bipartite graph of corresponding heaps. Moreover, if we have a morphism of heaps which is onto, we investigate the bipartite graph of the domain and the bipartite graph of the codomain. Furthremore, if we have two heaps, we find the relation between the bipartite graph of cross product of those two heaps and the bipartite graph of each component.

Acknowledgement Appreciation is extended to the reviewers and editors whose constructive feedback improved the quality and clarity of this article.

Conflicts of Interest The authors declare no conflict of interest.

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