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Full automorphism group of commuting and non-commuting graph of dihedral and symmetric groups

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Abstract: An automorphism of finite graph G is a permutation on its vertex set that conserves adjacency. The set of all automorphism of G is a group under composition of function. This group is called the full automorphism group of G. Study on the full automorphism group is an interesting topic because most graphs have only the trivial automorphism and many special graphs have many automorphisms. One of the special graphs is graph that associated with group. The result of this paper is the full automorphism groups of commuting and non-commuting graph of non-abelian finite group, especially on dihedral and symmetric groups, according to the choice of their subgroups.

1. Introduction

Graph in this paper is finite graph without loop and multiple edges. Let $G(V, E)$ be a graph with vertex set V and edge set E. An automorphism of graph G is a bijective function φ from V onto V such that $uv \in E$ if and only if $\varphi(u)\varphi(v) \in E$, for all $u, v \in V$ [1]. In other word, an automorphism of graph G is a permutation on V that conserves adjacency [2][3]. Trivial automorphism id from V onto V defined by $id(v) = v$ is an automorphism for every graph G. Because an automorphism is a bijection on V, it has an inverse that also an automorphism on V. Thus, the set of all automorphism of graph G will be a group together with function composition operation [4]. We denoted this group by $Aut(G)$ and named the full automorphism group of graph G[5].

More generally, an isomorphism from graph G to graph H is a bijective function φ from $V(G)$ onto $V(H)$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$, for all $u, v \in V(G)$. We called that G and H are isomorphic and denoted by $G \cong H$ if there exists an isomorphism from G to H [6]. If G and H are isomorphic, then $Aut(G)$ is isomorphic to $Aut(H)$ [4].

Most graphs have only trivial automorphism, but many special graphs have more automorphism other than trivial automorphism. Therefore, the study of full automorphism group of many special graphs has become an interesting topic and has been done [1][3][4][7-26]. In this paper, we address our study to the full automorphism group of the commuting and non-commuting graph of dihedral and symmetric groups.

2. Basic Definitions and Notations

Let G be a finite non-commutative group and $Z(G)$ is its center. Let X be any subset of G. The commuting graph $C(G, X)$ of G is a graph that contains X as its vertex set and two different vertices x and y will be adjacent in $C(G, X)$ if they are commute in G [27-29]. Let Y be any subset of $G \backslash Z(G)$. The non-commuting graph $\Gamma(G, Y)$ of G is a graph that contains Y as its vertex set and two different vertices x and y will be adjacent in $\Gamma(G, Y)$ if they are not commute in G. In the case $Y = G \backslash Z(G)$, then this definition is equal to definition that given by Abdollahi *et al* [30].

The dihedral group is the group of all symmetries on a regular polygon with n sides [31] and in this paper we denoted it by D_{2n}, where n is positive integer and $n \geq 3$. The dihedral group D_{2n} has order $2n$, that is $D_{2n} = \{1, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}$ with $s^2 = r^n = 1$.

The symmetric group on n elements is the group of all permutations on set $A = \{1, 2, 3, \dots, n\}$ and we denoted it by S_n. The symmetric group S_n has order $n!$ Let $a_1, a_2, a_3, \dots, a_r$ $(r \leq n)$ are distinct elements *in* A. Then $(a_1 a_2 a_3 \cdots a_r)$ denotes the permutation that maps a_1 to a_2, a_2 to a_3, a_3 to a_4, …, a_{r-1} to a_r, a_r to a_1 and maps every other element of A onto itself. $(a_1 a_2 a_3 \cdots a_r)$ is called an r-cycle or a cycle of length r. Transposition is a 2-cycle or cycle of length 2. During this paper, we consider S_n for $n \geq 3$.

In order to avoid distraction in symbolization, we will use $K_{1,n}$ to notate star graph of order $(n + 1)$. Complete graph and path graph with n vertices are denoted by K_n and P_n, respectively. Graph that consists of t copies of complete graph K_n with a vertex in common is called the windmill graph and will be denoted by K_n^t.

The following is previous result on the full automorphism group of graphs.

PROPOSITION 1: (a) For any graph G, $Aut(G) \cong Aut(\bar{G})$, (b) $Aut(nG) \cong S_n(Aut(G))$, and (c) $Aut(K_n) \cong S_n$ [3].

3. Results

We start the discussion of full automorphism group on dihedral group D_{2n} $(n \geq 3)$ first and then on symmetric group S_n $(n \geq 3)$.

THEOREM 1. Let $X = \{r^i \mid 1 \leq i \leq n\}$ be a subset of D_{2n}. Then, $Aut(C(D_{2n}, X)) \cong S_n$.

PROOF. Since $r^i r^j = r^j r^i$ in D_{2n}, for all $r^i, r^j \in X$, then the commuting graph $C(D_{2n}, X)$ is isomorphic to complete graph K_n. According to Proposition 1(c), we have the proof. ∎

THEOREM 2. Let $X = \{sr^i \mid 1 \leq i \leq n\} \cup \{1\}$ be a subset of D_{2n}, where n is odd. Then, $Aut(C(D_{2n}, X)) \cong S_n$.

PROOF. For odd n, $sr^i sr^j \neq sr^j sr^i$ $(1 \leq i, j \leq n$ and $i \neq j)$ in D_{2n} and therefore sr^i and sr^j are not adjacent in $C(D_{2n}, X)$. Since 1 is identity element in D_{2n}, then 1 is adjacent to sr^i $(1 \leq i \leq n)$ in $C(D_{2n}, X)$. So, $C(D_{2n}, X)$ is isomorphic to star graph $K_{1,n}$. For star graph $K_{1,n}$, its automorphism group contains all permutation on n non-central vertices. We conclude that $Aut(C(D_{2n}, X)) \cong S_n$. ∎

From Theorem 2, we have the following corollary.

COROLLARY 1. Let $X = \{sr^i \mid 1 \leq i \leq n\}$ be a subset of D_{2n}, where n is odd positive integer and $n \geq 3$. Then, $Aut(C(D_{2n}, X)) \cong S_n$.

PROOF. According to the proof of Theorem 2, we obtain that $(C(D_{2n}, X))$ is isomorphic to $\overline{K_n}$. From Proposition 1, we conclude that $Aut(C(D_{2n}, X)) \cong S_n$. ∎

THEOREM 3. Let $X = \{sr^i \mid 1 \leq i \leq n\} \cup \{1\}$ be a subset of D_{2n}, where n is even. Then, $Aut(C(D_{2n}, X))$ is isomorphic to $Aut\left(K_3^{n/2}\right)$.

PROOF. For even n, we have $Z(D_{2n}) = \{1, r^{n/2}\}$. So, 1 is commute to all element of X. And, we can see that $sr^i sr^{n/2+i} = sr^i sr^{n/2} r^i = sr^i r^{n/2} sr^i = sr^{n/2+i} sr^i$, for $i = 0, 1, 2, \dots, (n/2) - 1$. It means that sr^i and $sr^{n/2+i}$ $(0 \leq i < (n/2) - 1)$ are commute to each other. But, sr^i is not commute to sr^j, for $i \neq j$ and $j \neq n/2$ $(1 \leq i, j \leq n)$. Therefore, 1, sr^i, and $sr^{n/2+i}$ will be adjacent in $C(D_{2n}, X)$.

Thus, $C(D_{2n}, X)$ consists of (n/2) copies of the complete graph K_3 with 1 as a common vertex. Therefore, $Aut(C(D_{2n}, X))$ is isomorphic to $Aut\left(K_3^{n/2}\right)$. ■

From Theorem 3, because $Z(D_{2n}) = \{1, r^{n/2}\}$ for even n and $n \geq 3$, we will obtain the similar result for $X_1 = \{sr^i | 1 \leq i \leq n\} \cup \{r^{n/2}\}$.

COROLLARY 2. Let $X = \{sr^i | 1 \leq i \leq n\}$ be a subset of D_{2n}, where n is even. Then, $Aut(C(D_{2n}, X))$ is isomorphic to $S_{n/2}(S_2)$.

PROOF. According to the proof of Theorem 3, we have $C(D_{2n}, X)$ consists of (n/2) copies of the complete graph K_2. By Proposition 1(b), we conclude that $S_{n/2}(Aut(K_2)$. By Proposition 1(a), we complete the proof. ■

The following results are the full automorphism group of non-commuting graph on dihedral group $\Gamma(D_{2n}, Y)$ for some subset Y of $D_{2n} \backslash Z(D_{2n})$.

THEOREM 4. Let $Y = \{sr^i, r, r^2, ..., r^{n-1}\}$ be a subset of $D_{2n} \backslash Z(D_{2n})$ for some i ($1 \leq i \leq n$) where n is odd positive integer. Then, $Aut(\Gamma(D_{2n}, Y)) \cong S_{n-1}$.

PROOF. For odd n, $Z(D_{2n}) = \{1\}$. Since $r^i r^j = r^j r^i$ in D_{2n}, for all $r^i, r^j \in Y$, then $r^i$and r^j are not adjacent in $\Gamma(D_{2n}, Y)$. And, because sr^i is not commute to all r^j ($1 \leq j \leq n-1$), we have sr^i is adjacent to all r^j in $\Gamma(D_{2n}, Y)$. Thus, $\Gamma(D_{2n}, Y)$ is isomorphic to $K_{1,n-1}$ with sr^i as a central vertex. Therefore, $Aut(\Gamma(D_{2n}, Y)) \cong S_{n-1}$. ■

THEOREM 5. Let $Y = \{sr^i, r, r^2, ..., r^{n/2-1}, r^{n/2+1}, ..., r^{n-1}\}$ be a subset of $D_{2n} \backslash Z(D_{2n})$ for some i, $i \neq n/2$, and n is even positive integer. Then, $Aut(\Gamma(D_{2n}, Y)) \cong S_{n-2}$.

PROOF. For even n, $Z(D_{2n}) = \{1, r^{n/2}\}$. Since $r^i r^j = r^j r^i$ in D_{2n}, for all $r^i, r^j \in Y$, then $r^i$and r^j are not adjacent in $\Gamma(D_{2n}, Y)$. And, because sr^i is not commute to all elements r^j of Y, we have sr^i is adjacent to all r^j in $\Gamma(D_{2n}, Y)$. Thus, $\Gamma(D_{2n}, Y)$ is isomorphic to the star graph $K_{1,n-2}$ with sr^i as the central vertex. Therefore, $t(\Gamma(D_{2n}, Y)) \cong S_{n-2}$. ■

THEOREM 6. Let $Y = \{sr^i | 1 \leq i \leq n\}$ be a subset of $D_{2n} \backslash Z(D_{2n})$ where n is odd positive integer. Then, $Aut(\Gamma(D_{2n}, Y)) \cong S_n$.

PROOF. For odd n, $sr^i sr^j \neq sr^j sr^i$ ($0 \leq i, j < n$) in D_{2n} and therefore sr^i and sr^j are adjacent in $\Gamma(D_{2n}, Y)$. So, $\Gamma(D_{2n}, Y)$ is isomorphic to K_n. ■

And now, we present the full automorphism group on symmetric group S_n for several subsets X of S_n. The main results for commuting graph of S_n as the following theorems.

THEOREM 7. Let X be a subset of S_n that contains identity element 1 and all single transposition ($1a$) in S_n where $a \in \{2, 3, ..., n\}$. Then, $Aut(C(S_n, X)) \cong S_{n-1}$.

PROOF. Identity element 1 is commute to all element of S_n and $(1a)(1b) = (1ba) \neq (1ab) = (1b)(1a)$ for all $(1a)$ and $(1b)$ in X, where $a \neq b$. Therefore, $C(D_{2n}, X)$ is isomorphic to star graph $K_{1,n-1}$with 1 as its central vertex. Thus, $t(C(S_n, X)) \cong S_{n-1}$. ■

COROLLARY 3. Let X be a subset of S_n that contains all single transposition ($1a$) in S_n where $a \in \{2, 3, ..., n\}$. Then, $t(C(S_n, X)) \cong S_{n-1}$.

PROOF. According to the proof of Theorem 7, we have $C(D_{2n}, X)$ is isomorphic $\overline{K_{n-1}}$. By Proposition 1(a) and then Proposition 1(c), we get the desired result. ■

THEOREM 8. Let X be a subset of S_n that contains identity element 1 and all single 3-cycle in S_n. Then, $Aut(C(S_n, X))$ is isomorphic to $Aut\left(K_3^{(|X|-1)/2}\right)$.

PROOF. For any element (abc) of X, we have $(cba) \in X$ and $(abc)(cba) = 1 = (cba)(abc)$. Thus, ($abc$) and ($cba$) are commute in S_n. So, (abc) and (cba) are adjacent in $C(S_n, X)$. Because the identity element 1 is contained in X, then 1, (abc) and (bca) are adjacent in $C(S_n, X)$. Hence, we have $C(S_n, X)$ consisting of $(|X| - 1)/2$ copies of the complete graph K_3 with 1 as a common vertex. Thus, $C(S_n, X)$ is isomorphic to $K_3^{(|X|-1)/2}$. ■

COROLLARY 4. Let X be a subset of S_n that contains all single 3-cycle in S_n. Then, $Aut(C(S_n, X))$ is isomorphic to $S_{(|X|-1)/2}(S_2)$.

PROOF. From the proof of Theorem 8, $C(S_n, X)$ consisting of $(|X| - 1)/2$ copies of the complete graph K_2. By Proposition 1(b), it completes the desired proof. ■

The symmetric group S_n ($n \geq 3$) has a trivial center, that is $Z(S_n) = \{1\}$. Our main results for full automorphism group of non-commuting graph of symmetric group as follows.

THEOREM 9. Let Y be a subset of $S_n \backslash Z(S_n)$ that contains all single transpositions $(1a)$ in S_n, where $a \in \{2, 3, ..., n\}$. Then, $Aut\big(\Gamma(S_n, Y)\big) \cong S_{n-1}$.

PROOF. Because $(1a)(1b) = (1ba) \neq (1ab) = (1b)(1a)$ for all $(1a)$ and $(1b)$ in X, where $a \neq b$, then $\Gamma(S_n, Y)$ is a complete graph of order $(n - 1)$. Thus, $Aut\big(\Gamma(S_n, Y)\big) \cong S_{n-1}$. ■

THEOREM 10. Let Y be a subset of $S_n \backslash Z(S_n)$ that contains all single transpositions $(a(a+1))$ in S_n, where $a \in \{1, 2, 3, ..., n\text{-}1\}$. Then, $Aut(\Gamma(S_n, Y))$ is isomorphic to $Aut(P_{n-1})$.

PROOF. Let $x = (a(a+1))$ and $y = ((a+1)(a+2))$ are element of Y. We see that xy = $(a(a+1))(a+2))$

and $yx = (a(a+2)(a+1))$. So, x and y are not commute and will be adjacent in $\Gamma(S_n, Y)$. In other hand, for $(c(c+1))$ and $(d(d+1))$, where $d \neq c + 1$, we have $(c(c+1))(d(d+1)) = (d(d+1))(c(c+1))$. Hence, $(c(c+1))$ and $(d(d+1))$, where $d \neq c + 1$, are commute to each other and will be not adjacent in $\Gamma(S_n, Y)$. So, $\Gamma(S_n, Y)$ is a path graph P_{n-1}. Therefore, $t\big(\Gamma(S_n, Y)\big) \cong Aut(P_{n-1})$. ■

4. Conclusion

We have determined the full automorphism group on commuting and non-commuting graph of dihedral and symmetric groups for their several subsets. Further studies can be performed on other graphs of dihedral and symmetric groups such as conjugate graph and inverse graph.

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