The Adjacency Spectrum of Subgroup Graphs of Dihedral Group

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The Adjacency Spectrum of Subgroup Graphs of Dihedral Group

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Abstract. Research topics on graph associated with a group are subject of much investigation today as well as research topics on the spectra of graph. However, research on the adjacency spectrum of subgroup graph of dihedral group is not reported yet. By this reason, we determined the adjacency spectrum of subgroup graphs of dihedral group and their complements.

1. Introduction

Let G be a graph of order p and $A(G)$ be adjacency matrix of G. The characteristic polynomial of $A(G)$ is $p(\lambda) = \det(A(G) - \lambda I)$. The roots of $p(\lambda) = 0$ are called eigenvalues of $A(G)$. If all eigenvalues of $A(G)$ are integer then G is called integral [30]. The collection of all distinct eigenvalues of $A(G)$ together with their multiplicities is called the adjacency spectrum of G [10]. The adjacency spectrum of graph is also called A-spectrum of G [16].

Previous research on adjacency spectrum of several graphs has been reported. The adjacency spectrum of complete multipartite graph [5], corona $G_1 \circ K_{m1,m2}$ [14], traffic network [24], lollipop graphs [20], graph Gl [32], neighborhood corona of graph [19], coalescence of complete graphs [22] and graphs with pockets [15] have been observed and determined.

Graph can also be obtained from a finite group, for example commuting graph [7][13][25][29][31], non-commuting graph [1], conjugate graph [18], Cayley graph [21][26][27] and invers graph [6]. Anderson, Fasteen and Lagrange [8] introduced the concept of graph obtained from the group called subgroup graph. For a finite group G and a subgroup H of G, the subgroup graph $\Gamma_{\mathrm{H}}(\mathrm{G})$ of G is a directed graph with vertex set G. Kakeri and Erfanian [23] explained that if is normal subgroup in G , then the subgroup $\Gamma_{\mathrm{H}}(\mathrm{G})$ graph of G is an undirected graph. This implies that the complement of subgroup graph $\Gamma_{\mathrm{H}}(\mathrm{G})$ is also undirected graph.

Several research on the adjacency spectrum of a graph obtained from a group have been reported. Abdussakir, Elvierayani and Nafisah [4] investigated the adjacency spectrum of commuting and non-commuting graph of dihedral group. Abdussakir [2] also investigated the adjacency spectrum of conjugate graph of dihedral group. Because the adjacency spectrum of subgroup graphs of dihedral group has not been determined yet, we conducted this research. The main objective of this research is to determine the adjacency spectrum of subgroup graphs of dihedral group and their complements. The adjacency energy of these graphs also discussed in this paper.

2. Literature Review

Graph G is a pair $(V(G), E(G))$ where $V(G)$ is finite non-empty set of objects called vertex and $E(G)$ is the set (possibly empty) of unordered pairs of different vertexs in $V(G)$ called edges. The number of elements in $V(G)$ is called the order of G and is denoted by p, and the number of elements in $E(G)$ is called the size of G and is denoted by q. We denoted uv for (u, v) in $E(G)$. The complement of graph G, written by $\bar{G}$, is a graph with vertex set $V(G)$ such that two vertices are adjacent in $\bar{G}$ if and only if those two vertices are not adjacent in G. Thus, $V(\bar{G}) = V(G)$ and $uv \notin E(G)$ if and only if $uv \notin E(G)$[3].

A graph G is said to be a complete graph if any two distinct vertices in graph G are adjacent. A complete graph of order n is denoted by K_n. A graph G is bipartite if V(G) can be partitioned into two sets (called partition sets) V_1 and V_2 so that every edge of G joins a vertex of $V_1$and a vertex of V_2. A graph G is complete bipartite if G is bipartite and for every $u \in V_1$ and $u \in V_2$, then $uv \in E(G)$. If $|V_1| = m$ and $|V_2| = n$, then this complete bipartite graph is denoted by $K_{m,n}$. For more general, let k be positive integer and k > 1. A graph G is said to be a complete multipartite if V(G) can be partitioned into k partition sets $V_1, V_2, \dots, V_k$ so that every edge of G joins vertices in different partition set and two vertices are adjacent if and only if the vertices belong to distinct partition sets [12]. If $|V_i| = m_i$ for 1 □ i □ k, then this complete multipartite graph is denoted by $K_{m_1,m_2,\dots,m_k}$.

Suppose G graph with order $p(p \geq 1)$ and size q and vertex set $V(G) = \{v_1, v_2, \dots, v_p\}$. The adjacency matrix of graph G, denoted by A(G), is the $(p \times p)$ matrix $A(G) = [a_{ij}]$ where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$ if $v_i v_j \notin E(G)$ (Chartrand et al. 2016). The characteristic polynomial of A(G) is $p(\lambda) = \det(A(G) - \lambda I)$, where I is $(p \times p)$ identity matrix. The roots of $p(\lambda) = 0$ are called eigenvalues of A(G) [22]. Because A(G) is a real symmetric matrix then all eigenvalues of A(G) are real [11] and can be ordered by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. The adjacency energy of G is defined by $E_A(G) = \sum_{i=1}^{p} |\lambda_i|$ [9][28]. Let $\lambda_{1'} > \lambda_{2'} > \cdots > \lambda_{n'}$ are distinct eigenvalues of A(G) and $m(\lambda_{1'}), m(\lambda_{2'}), \dots, m(\lambda_{n'})$, are corresponding multiplicities of $(\lambda_{1'})$. According to Yin [32], the adjacency spectrum of G is defined by

$$spec(A(G) = \begin{bmatrix} \lambda_{1'} & \lambda_{2'} & \cdots & \lambda_{n'} \\ m(\lambda_{1'}) & m(\lambda_{2'}) & \cdots & m(\lambda_{n'}) \end{bmatrix} \quad (1)$$

Let G be a finite group and H a subgroup of G. Let $\Gamma_H(G)$ be the directed graph with vertex set G such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$. Furthermore, if $xy \in H$ and $yx \in H$ for $x, y \in G$ and $x \neq y$ then x and y will be regarded as being connected by a single undirected edge. Thus, we get the graph $\Gamma_H(G)$ which has no loops or multiple edges. The graph $\Gamma_H(G)$ is called the subgroup graph of G [8]. Kakeri and Erfanian [23] explain that the subgroup graph $\Gamma_H(G)$ is clear of its existence when H is the normal subgroup of G. If H normal subgroup of G, $xy \in H$ then will imply that $yx \in H$. Thus, $\Gamma_H(G)$ and its complement is an undirected simple graph when H is normal subgroup of G.

The dihedral group is a group of the sets of n-symmetry, denoted D_{2n}, for any positive integers n and $n \geq 3$. Since dihedral groups will be used extensively along this paper, it will take some notation and some computation that can simplify the next calculation and help observe D_{2n} as abstract group, that is:

(1) $1, r, r^2, \dots, r^{n-1}$ are distinct and $|r| = n$.
(2) $|s| = 2$
(3) $s \neq r^i$ for all i.
(4) $sr^i \neq sr^j$ for all $0 \leq i, j \leq n-1$ with $i \neq j$. Then, $D_{2n} = \{1, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}$ and each element can be written $s^k r^i$ for $k = 0$ or 1 and $0 \leq i \leq n-1$.
(5) $sr = r^{-1}s$.
(6) $sr^i \neq sr^j$ for all $0 \leq i \leq n$ [17]

For odd *n,* all normal subgroups of D_{2n} are $\langle 1\rangle, \langle r^d\rangle$ for all *d* dividing *n* and D_{2n} itself. For even *n*, all normal subgroups of are $\langle 1\rangle, \langle r^d\rangle$ for all *d* dividing $n, \langle r^2, s\rangle, \langle r^2, rs\rangle$ and D_{2n} itself.

3. Results and Discussion

The followings are our main results on the adjacency spectrum of subgroup graph of dihedral group and their complement. We also present their adjacency energy as additional results.

Theorem 3.1: *The adjacency spectrum of subgroup graph* $\Gamma_{\langle r\rangle}(D_{2n})$ *of dihedral group* D_{2n} *for positive integer* n *and* $n \geq 3$ *is*

$$spec\left(A\left(\Gamma_{\langle r\rangle}(D_{2n})\right)\right) = \begin{bmatrix} n-1 & -1 \\ 2 & 2n-2 \end{bmatrix}$$

Proof: For $n \geq 3$, we have that normal subgroup $\langle r\rangle$ of dihedral group D_{2n} is $\langle r\rangle = \{1, r, r^2, \ldots, r^{n-1}\}$. According to the definition of subgroup graph, we have the subgroup graph $\Gamma_{\langle r\rangle}(D_{2n})$ is disconnected graph with two components. The first component is complete graphs K_n with vertex set $\{1, r, r^2, \ldots, r^{n-1}\}$ and the second component is complete graphs K_n with vertex set $\{s, sr, sr^2, \ldots, sr^{n-1}\}$. Hence, the adjacency matrix of $\Gamma_{\langle r\rangle}(D_{2n})$ is

$$A\left(\Gamma_{\langle r\rangle}(D_{2n})\right) = \begin{bmatrix} Q & O \\ O & Q \end{bmatrix}$$

where $Q = [q_{ij}]$ is $(n \times n)$ matrix where q_{ij} for $i \neq j$ and $q_{ij} = 0$ for others and O is $(n \times n)$ zero matrix. The characteristic polynomial $A\left(\Gamma_{\langle r\rangle}(D_{2n})\right)$ is equal to $\det\left(A\left(\Gamma_{\langle r\rangle}(D_{2n})\right) - \lambda I\right)$. Performing Gaussian elimination on $A\left(\Gamma_{\langle r\rangle}(D_{2n})\right) - \lambda I$, we get the following upper triangular matrix

$$\begin{bmatrix} (-\lambda)^2 & \cdots & \cdots & \cdots & \cdots \\ 0 & -\frac{(\lambda-1)(\lambda+1)}{\lambda} & \cdots & \cdots & \cdots \\ 0 & 0 & -\frac{(\lambda-2)(\lambda+1)}{\lambda-1} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -\frac{\lambda-(n-1)(\lambda+1)}{\lambda-(n-2)} \\ 0 & 0 & 0 & \cdots & \end{bmatrix}$$

Accordingly, $\det\left(A\left(\Gamma_{\langle r\rangle}(D_{2n})\right) - \lambda I\right)$ is the multiplication of the main diagonal elements of this upper triangular matrix. Thus, the characteristic polynomial of $A\left(\Gamma_{\langle r\rangle}(D_{2n})\right)$ is

$$p(\lambda) = \left(\lambda - (n-1)\right)^2 (\lambda+1)^{2n-2}$$

Let $p(\lambda) = 0$. We obtaine the eigenvalues $\lambda_1 = (n-1)$ and $\lambda_2 = -1$ and their multiplicity $m(\lambda_1) = 2$ and $m(\lambda_2) = 2n-2$, respectively. Thus, the adjacency spectrum of subgroup graph $\Gamma_{\langle r\rangle}(D_{2n})$ for positive integer n and $n \geq 3$ is

$$spec\left(A\left(\Gamma_{\langle r\rangle}(D_{2n})\right)\right) = \begin{bmatrix} n-1 & -1 \\ 2 & 2n-2 \end{bmatrix}$$

Corollary 3.1: *The adjacency energy of subgroup graph* $\Gamma_{\langle r\rangle}(D_{2n})$ *of dihedral group* D_{2n} *for positive integer* n *and* $n \geq 3$ *is*

$$E_A\left(\Gamma_{\langle r\rangle}(D_{2n})\right) = 4(n-1).$$

Proof: From Theorem 3.1, we have $E_A\left(\Gamma_{\langle r\rangle}(D_{2n})\right) = 2(n-1) + 2(n-1)(1) = 4(n-1)$.

Theorem 3.2: *The adjacency spectrum of complement of subgroup graph* $\overline{\Gamma_{\langle r\rangle}(D_{2n})}$ *of dihedral group* D_{2n} *for positive integer* n *and* $n \geq 3$ *is*

$$spec\left(A\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right)\right) = \begin{bmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{bmatrix}$$

Proof: Based on the proof of Theorem 3.1, the complement of subgroup graph $\overline{\Gamma_{\langle r\rangle}(D_{2n})}$ is a complete bipartite graph $K_{n,n}$ with the partition sets $V_1 = \{1, r, r^2, \dots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \dots, sr^{n-1}\}$. So, the adjacency matrix of $\overline{\Gamma_{\langle r\rangle}(D_{2n})}$ is as follows.

$$A\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right) = \begin{bmatrix} O & T \\ T & O \end{bmatrix}$$

where T is $(n \times n)$ ones matrix and O is $(n \times n)$ zero matrix. Using Gaussian elimination on $A\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right) - \lambda I$ we obtaine the following upper triangular matrix.

$$\begin{bmatrix} -\lambda & \cdots & \cdots & \cdots & \cdots \\ 0 & -\lambda & \cdots & \cdots & \cdots \\ 0 & 0 & -\lambda & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & -\dfrac{\lambda(\lambda^2 - n^2)}{\lambda^2 - (n^2 - n)} \end{bmatrix}$$

Then, the characteristic polynomial of $A\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right)$ is

$$p(\lambda) = (\lambda - n)\lambda^{2n-2}(\lambda + n)$$

By setting $p(\lambda) = 0$, we obtaine the eigenvalues $\lambda_1 = n, \lambda_2 = 0$ and $\lambda_3 = -n$ and their multiplicity $m(\lambda_1) = 1$, $m(\lambda_2) = 2n - 2$ and $m(\lambda_3) = 1$, respectively. Thus, the adjacency spectrum of subgroup graph $\overline{\Gamma_{\langle r\rangle}(D_{2n})}$ for positive integer n and $n \geq 3$ is

$$spec\left(A\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right)\right) = \begin{bmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{bmatrix}$$

Corollary 3.2: *The adjacency energy of subgroup graph $\overline{\Gamma_{\langle r\rangle}(D_{2n})}$ of dihedral group D_{2n} for positive integer n and $n \geq 3$ is*

$$E_A\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right) = 2n$$

Proof: From Theorem 3.2, we have $E_A\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right) = 1|n| + 2(n-1)|0| + 1|-n| = 2n$

Theorem 3.3: *The adjacency spectrum of subgroup graph $\Gamma_{\langle r^2\rangle}(D_{2n})$ of dihedral group D_{2n} for even n and $n \geq 4$ is*

$$spec\left(A\left(\Gamma_{\langle r^2\rangle}(D_{2n})\right)\right) = \begin{bmatrix} \dfrac{n-2}{2} & -1 \\ 4 & 2n-4 \end{bmatrix}$$

Proof: For even n and $n \geq 4$, the subgroup $\langle r^2\rangle$ of dihedral group D_{2n} is $\langle r^2\rangle = \{1, r^2, \dots, r^{n-2}\}$. According to the definition of subgroup graph, then the subgroup graph $\Gamma_{\langle r^2\rangle}(D_{2n})$ is disconnected graph that consists of four complete graphs $K_{n/2}$ as its components. The vertex set of these four components are $\{1, r^2, \dots, r^{n-2}\}, \{r, r^3, \dots, r^{n-1}\}, \{s, sr^2, \dots, sr^{n-2}\}$ .So, the adjacency matrix of $\Gamma_{\langle r^2\rangle}(D_{2n})$ is as follows.

$$A\left(\Gamma_{\langle r^2\rangle}(D_{2n})\right) = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \ddots & 0 & 1 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & & 0 & 0 & 1 & 0 & & 0 \end{bmatrix}$$

Using Gaussian elimination on $A\left(\Gamma_{\langle r^2\rangle}(D_{2n})\right) - \lambda I$ we obtaine the following upper triangular matrix.

$$\begin{bmatrix} -\lambda & \cdots & \cdots & \cdots & \cdots \\ 0 & -\lambda & -\dfrac{(\lambda-1)(\lambda+1)}{\lambda} & \cdots & \vdots \\ 0 & 0 & \vdots & \ddots & -\dfrac{(\lambda-\frac{n-2}{2})(\lambda+1)}{\lambda-\frac{n-4}{2}} \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & \end{bmatrix}$$

Thus, the characteristic polynomial of $A\left(\Gamma_{\langle r^2\rangle}(D_{2n})\right)$ is

$$p(\lambda)=(-1)^{\frac{n}{2}}\left(\lambda-\frac{n-2}{2}\right)^4(\lambda+1)^{2n-4}$$

By setting $p(\lambda)=0$, we obtain the eigenvalues $\lambda_1=\frac{n-2}{2}$ and $\lambda_2=-1$ and their multiplicity $m(\lambda_1)=4$ and $m(\lambda_2)=2n-4$, respectively. Thus, the adjacency spectrum of subgroup graph $\Gamma_{\langle r^2\rangle}(D_{2n})$ for even n and $n\geq 4$ Is

$$spec\left(A\left(\Gamma_{\langle r^2\rangle}(D_{2n})\right)\right)=\begin{bmatrix}\frac{n-2}{2} & -1\\ 4 & 2n-4\end{bmatrix}$$

Corollary 3.3: *The adjacency energy of subgroup graph $\Gamma_{\langle r^2\rangle}(D_{2n})$ of dihedral group D_{2n} for even n and $n\geq 4$ is*

$$E_A\left(\Gamma_{\langle r^2\rangle}(D_{2n})\right)=4(n-2)$$

Proof: From Theorem 3.3, we have $E_A\left(\Gamma_{\langle r^2\rangle}(D_{2n})\right)=4\left|\frac{n-2}{2}\right|+2(n-2)|-1|=4(n-2)$

Theorem 3.4: *The adjacency spectrum of complement of subgroup graph $\Gamma_{\langle r^2\rangle}(D_{2n})$ of dihedral group D_{2n} for even n and $n\geq 4$ is*

$$spec\left(A\left(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}\right)\right)=\begin{bmatrix}\frac{3n}{2} & 0 & \frac{n}{2}\\ 1 & 2n-4 & 3\end{bmatrix}$$

Proof: From the proof of Theorem 3.3, then the complement of subgroup graph $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ of dihedral group D_{2n} is complete multipartite graph $K_{n/2,n/2,n/2,n/2}$. Therefore, the adjacency matrix $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ is as follows.

$$A\left(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}\right)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1\\ 1 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1\\ 0 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1\\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 0 & \cdots & 1\\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & \cdots & 0\\ 1 & 1 & 1 & \ddots & 1 & 0 & 0 & 0 & \ddots & 1\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 1 & 1 & 1 & & 1 & 0 & 1 & 0 & & 0 \end{bmatrix}$$

Using Gaussian elimination on $A\left(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}\right)-\lambda I$ we obtaine the following upper triangular matrix.

$$\begin{bmatrix} -\lambda & \cdots & \cdots & \cdots & \cdots \\ 0 & -\dfrac{\lambda^2-1}{\lambda} & \cdots & \cdots & \vdots \\ 0 & 0 & -\dfrac{\lambda(\lambda^2-2)}{\lambda^2-1} & \ddots & -\dfrac{(\lambda-\frac{3n}{2})(\lambda+\frac{n}{2})}{\lambda^2-n\lambda-\left(\frac{3n^2-6n}{4}\right)} \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \end{bmatrix}$$

Thus, the characteristic polynomial of $A\left(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}\right)$ is

$$p(\lambda)=\left(\lambda-\frac{3n}{2}\right)\lambda^{2n-4}\left(\lambda+\frac{n}{2}\right)^{3}$$

By setting $p(\lambda)=0$, we obtaine $\lambda_1=\frac{3n}{2},\lambda_2=0$ and $\lambda_3=-\frac{n}{2}$ and the multiplicity $m(\lambda_1)=1, m(\lambda_2)=2n-4$ and $m(\lambda_3)=3$. So, the adjacency spectrum of subgroup graph of $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ for even n and $n\geq 4$ is

$$spec\left(A\left(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}\right)\right)=\begin{bmatrix}\frac{3n}{2} & 0 & \frac{n}{2}\\ 1 & 2n-4 & 3\end{bmatrix}$$

Corollary 3.4: *The adjacency energy of subgroup graph $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ of dihedral group D_{2n} for even n and $n\geq 4$ is*

$$E_A\left(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}\right)=3n$$

Proof: From Theorem 3.4, we have $E_A\left(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}\right)=1\left|\frac{3n}{2}\right|+2(n-2)|0|+3\left|\frac{n}{2}\right|=3n$

Theorem 3.5: *The adjacency spectrum of subgroup graph $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ of dihedral group D_{2n} for even n and $n\geq 4$ is*

$$spec\left(A\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right)\right)=\begin{bmatrix}n-1 & -1\\ 2 & 2n-2\end{bmatrix}$$

Proof: For even n and $n\geq 4$, the subgroup $\langle r^2,s\rangle$ of dihedral group D_{2n} is $\langle r^2,s\rangle=\{1,r^2,\ldots,r^{n-2},s,sr^2,\ldots,sr^{n-2}\}$. Therefore, the subgroup graph $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ of two complete graphs K_n as its component. The first component with vertex set $\{1,r^2,\ldots,s,sr^2,\ldots\}$ and the second with vertex set $\{r,\ldots,r^{n-1},sr,\ldots,sr^{n-1}\}$. And then, the adjacency matrix $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ is as follows.

$$A\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right)=\begin{bmatrix}0 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 & \cdots & 1\\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 1\\ 1 & 0 & 1 & \vdots & 0 & 0 & 0 & 1 & \vdots & 0\\ 0 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1\\ 1 & 0 & 1 & \ddots & 0 & 1 & 0 & 0 & \ddots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 1 & 0 & & 1 & 0 & 1 & 0 & & 0\end{bmatrix}$$

Using Gaussian elimination on $A\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right)-\lambda I$ we obtaine the following upper triangular matrix.

$$\begin{bmatrix}-\lambda & \cdots & \cdots & \cdots & \cdots\\ 0 & -\lambda & \cdots & \cdots & \cdots\\ 0 & 0 & -\frac{(\lambda-1)(\lambda+1)}{\lambda} & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -\frac{(\lambda-(n-1))(\lambda+1)}{\lambda-(n-2)}\end{bmatrix}$$

Thus, the characteristic polynomial $A\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right)$ is

$$p(\lambda)=\left(\lambda-(n-1)\right)^2(\lambda+1)^{2n-2}$$

By setting $p(\lambda)=0$, we obtaine the eigenvalues $\lambda_1=n-1$ and $\lambda_2=-1$ and their multiplicity $m(\lambda_1)=2$ and$m(\lambda_2)=2n-2$, respectively. Therefore, the adjacency spectrum of subgroup graph $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ for even n and $n\geq 4$ is

$$spec\left(A\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right)\right)=\begin{bmatrix}n-1 & -1\\ 2 & 2n-2\end{bmatrix}$$

Corollary 3.5: *The adjacency energy of subgroup graph $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ of dihedral group D_{2n} for even n and $n\geq 4$ is*

$$E_A\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right) = 4(n-1)$$

Proof: From Theorem 3.5, we have $E_A\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right) = 4|n-1| + 2(n-1)|-1| = 4(n-1)$

Theorem 3.6: *The adjacency spectrum of complement of subgroup graph* $\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}$ *of dihedral group* D_{2n} *for even* n *and* $n \geq 4$ *is*

$$spec\left(A\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right)\right) = \begin{bmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{bmatrix}$$

Proof: According to the proof of Theorem 3.5, the complement of subgroup graph $\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}$ is a complete bipartite graph $K_{n,n}$ with partition sets $V_1 = \{1, r^2, \ldots, s, sr^2, \ldots\}$ and $V_2 = \{r, \ldots, r^{n-1}, sr, \ldots, sr^{n-1}$. Thus, the adjacency matrix of $\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}$ of is a follows.

$$A\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 1 & 0 & 1 & 0 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 0 & 1 & 0 & 1 & \ldots & 0 \\ 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \vdots & 1 & 0 & 1 & 0 & \vdots & 1 \\ 1 & 0 & 1 & \ldots & 0 & 1 & 0 & 1 & \ldots & 0 \\ 0 & 1 & 0 & \ddots & 1 & 0 & 1 & 0 & \ddots & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & 0 & 1 & & 0 & 1 & 0 & 1 & & 0 \end{bmatrix}$$

Using Gaussian elimination on $A\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right) - \lambda I$ we obtaine the following upper triangular matrix.

$$\begin{bmatrix} -\lambda & \lambda^2-1 & \cdots & \cdots & \cdots \\ 0 & -\frac{\lambda^2-1}{\lambda} & \cdots & \vdots & \cdots \\ 0 & 0 & -\frac{\lambda(\lambda^2-2)}{\lambda^2-1} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & -\frac{\lambda(\lambda^2-n^2)}{\lambda^2-(n^2-n)} \end{bmatrix}$$

Thus, the characteristic polynomial of $A\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right)$ is

$$p(\lambda) = (\lambda-n)\lambda^{2n-2}(\lambda+n)$$

By setting $p(\lambda) = 0$, we obtaine the eigenvalues $\lambda_1 = n, \lambda_2 = 0$ and $\lambda_3 = -n$ and their multiplicity $m(\lambda_1) = 1, m(\lambda_2) = 2n-2$ and $m(\lambda_3) = 1$, respectively. So, the adjacency spectrum of subgroup graph $\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}$ for even n and $n \geq 4$ is

$$spec\left(A\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right)\right) = \begin{bmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{bmatrix}$$

Corollary 3.6: *The adjacency energy of subgroup graph* $\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}$ *of dihedral group* D_{2n} *for even* n *and* $n \geq 4$ *is*

$$E_A\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right) = 2n$$

Proof: From Theorem 3.6, we have $E_A\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right) = 1|n| + 2(n-1)|0| + 1|-n| = 2n$

Theorem 3.7: *The adjacency spectrum of subgroup graph* $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ *of dihedral group* D_{2n} *for even* n *and* $n \geq 4$ *is*

$$spec\left(A\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right)\right) = \begin{bmatrix} n-1 & -1 \\ 2 & 2n-2 \end{bmatrix}$$

Proof: For even n and $n \geq 4$, the normal subgroup $\langle r^2, rs\rangle$ of dihedral group D_{2n} is $\langle r^2, rs\rangle = \{1, r^2, \ldots, r^{n-2}, sr, sr^3, \ldots, sr^{n-1}\}$. According to the definition of subgroup graph, then the subgroup graph $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ is disconnected graph with two complete graphs K_n as its component. The first component has the vertex set $\{1, r^2, \ldots, sr, \ldots, sr^{n-1}\}$ and the second component has the vertex set $\{r, \ldots, r^{n-1}, s, sr^2, \ldots$. So, the adjacency matrix of $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ is as follows.

$$A\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right)=\begin{bmatrix}0&0&1&\dots&0&0&1&0&\dots&1\\0&0&0&\dots&1&1&0&1&\dots&0\\1&0&0&\cdots&0&0&1&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&1&0&\cdots&0&1&0&1&\cdots&0\\0&1&0&\vdots&1&0&0&1&\vdots&0\\1&0&1&\dots&0&0&0&0&\dots&1\\0&1&0&\ddots&1&1&0&0&\ddots&0\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots&\vdots&\vdots&\dots&\vdots\\1&0&1&&0&0&1&0&&0\end{bmatrix}$$

Using Gaussian elimination on $A\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right)-\lambda I$ we obtaine the following upper triangular matrix.

$$\begin{bmatrix}-\lambda&\cdots&\cdots&&\cdots\\0&-\lambda&\cdots&\cdots&\cdots\\0&0&-\dfrac{(\lambda-1)(\lambda+1)}{\lambda}&\vdots&\vdots\\\vdots&\vdots&\vdots&\ddots&\\0&0&0&\cdots&-\dfrac{(\lambda-(n-1))(\lambda+1)}{\lambda-(n-2)}\end{bmatrix}$$

Thus, the characteristic polynomial $A\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right)$ is

$$p(\lambda)=\left(\lambda-(n-1)\right)^2(\lambda+1)^{2n-2}$$

By setting $p(\lambda)=0$, we obtaine the eigenvalues $\lambda_1=n-1$ and $\lambda_2=-1$ and their multiplicity $m(\lambda_1)=2$ and $m(\lambda_2)=2n-2$, respectively. Hence, the adjacency spectrum of subgroup graph $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ for even n and $n\geq 4$ is

$$spec\left(A\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right)\right)=\begin{bmatrix}n-1&-1\\2&2n-2\end{bmatrix}$$

Corollary 3.7: *The adjacency energy of subgroup graph* $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ *of dihedral group* D_{2n} *for even* n *and* $n\geq 4$ *is*

$$E_A\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right)=4(n-1)$$

Proof: From Theorem 3.7, we have $E_A\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right)=2|n-1|+2(n-1)|-1|=4(n-1)$

Theorem 3.8: *The adjacency spectrum of complement of subgroup graph* $\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}$ *of dihedral group* D_{2n} *for even* n *and* $n\geq 4$ *is*

$$spec\left(A\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right)\right)=\begin{bmatrix}n&0&-n\\1&2n-2&1\end{bmatrix}$$

Proof: Based on the proof of Theorem 3.7, the complement of subgroup graph $\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}$ is a complete bipartite graph $K_{n,n}$ with partition sets $V_1=\{1,r^2,\dots,sr,\dots,sr^{n-1}\}$ and $V_2=\{r,\dots,r^{n-1},s,sr^2,\dots,sr^{n-2}\}$. So, the adjacency matrix of $\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}$ is as follows.

$$A\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right)=\begin{bmatrix}0&1&0&\dots&1&1&0&1&\dots&0\\1&0&1&\dots&0&0&1&0&\dots&1\\0&1&0&\cdots&1&1&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&0&1&\cdots&0&0&1&0&\cdots&1\\1&0&1&\vdots&0&0&1&0&\vdots&1\\0&1&0&\dots&1&1&0&1&\dots&0\\1&0&1&\ddots&0&0&1&0&\ddots&1\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots&\vdots&\vdots&\dots&\vdots\\0&1&0&&1&1&0&1&&0\end{bmatrix}$$

Using Gaussian elimination on $A\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right)-\lambda I$ we obtaine the following upper triangular matrix.

$$\begin{bmatrix} -\lambda & \cdots & \cdots & \cdots & \cdots \\ 0 & -\frac{\lambda^2-1}{\lambda} & \cdots & \cdots & \cdots \\ 0 & 0 & -\frac{\lambda(\lambda^2-2)}{\lambda^2-1} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & -\frac{\lambda(\lambda^2-n^2)}{\lambda^2-(n^2-n)} \end{bmatrix}$$

Therefore, the characteristic polynomial of $A\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right)$ is

$$p(\lambda) = (\lambda - n)\lambda^{2n-2}(\lambda + n)$$

By setting $p(\lambda) = 0$, we obtaine the eigenvalues $\lambda_1 = n, \lambda_2 = 0$ and $\lambda_3 = -n$ and their multiplicity $m(\lambda_1) = 1, m(\lambda_2) = 2n - 2$ and $m(\lambda_3) = 1$, respectively. Hence, the adjacency spectrum of subgroup graph $\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}$ for even n and $n \geq 4$ is

$$spec\left(A\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right)\right) = \begin{bmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{bmatrix}$$

Corollary 3.8: *The adjacency energy of subgroup graph* $\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}$ *of dihedral group* D_{2n} *for even* n *and* $n \geq 4$ *is*

$$E_A\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right) = 2n$$

Proof: From Theorem 3.8, we have $E_A\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right) = 1|n| + 2(n-1)|0| + 1|-n| = 2n$

4. Conclusion

We have determined the adjacency spectrum and energy of subgroup graph of dihedral group D_{2n} and their complement for normal subgroups $\langle r\rangle$, $\langle r^2\rangle$, $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. We found that the subgroup graphs and their complements are integral for these normal subgroups. According to this result, we proposed the following open problem.

Problem: *Is the subgroup graph* $\Gamma_H(D_{2n})$ *of dihedral group* D_{2n} *integral for any normal subgroup H of* D_{2n} ?

For the further research, the adjacency spectrum of other graphs associated with dihedral group or other groups is needed to be observed.

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References

[1] Abdollahi A, Akbari S, and Maimani H R 2006 *J. Algebr.* **298** 468–492
[2] Abdussakir 2017 *Proceeding of Seminar Nasional Teknologi Informasi, Komunikasi dan Industri (SNTIKI)* **9** 670–674
[3] Abdussakir, Azizah N N, & Nofandika F F 2009 *Teori Graf* (Malang: UIN Press)
[4] Abdussakir, Elvierayani R R & Nafisah M 2017 *Cauchy-Jurnal Mat. Murni dan Apl.*, **4** 176–182
[5] Abdussakir, Ikawati D S & Sari F K 2013 *Research Report* Available online: http://repository.uin- malang.ac.id/1758/ last visit: 28.09.2018
[6] Alfuraidan M R & Zakariya Y F 2017 *Electron. J. Graph Theory Appl.* **5** 142–154
[7] Ali F, Salman M & Huang S 2016 *Commun. Algebra* **44** 2389–2401
[8] Anderson D F, Fasteen J, & Lagrange J D 2012 *Arab J. Math.* **1** 17–27
[9] Balakrishnan R 2004 *Linear Algebra Appl.* **387** 287–295
[10] Biggs N 1993 *Algebraic Graph Theory 2nd ed* (Cambridge: Cambridge University Press)
[11] Brouwer A E & Haemers W H 2012 *Graph Spectrum* In: *Spectra of graphs*. (New York: Springer)
[12] Chartrand G, Lesniak L, & Zhang P 2016 *Graphs and Digraphs 6th ed.* (Florida: CRC Press)
[13] Chelvam T T, Selvakumar K, & Raja S 2011 *J. Math. Comput. Sci.* **2** 402–406

[14] Cheng L & Huang Q 2011 *J. Xinjiang University (Natural Science Edition)* **2** 156-162
[15] Cui S Y & Tian G X 2017 *Appl. Math. Comput.* **315** 363–371
[16] Cvetković D & Simić S K 2010 *Linear Algebra Appl.* **432** 2257–2272
[17] Dummit D S & Foote R M 1991 *Abstract Algebra* (New Jersey: Prentice Hall)
[18] Erfanian A & Tolue B 2012 *Discret. Math. Algorithms Appl.* **04** 1–8
[19] Gopalapillai I 2011 *Kragujev. J. Math.* **35** 493–500
[20] Haemers W H, Liu X, & Zhang Y 2008 *Linear Algebra Appl.* **428** 2415–2423
[21] Hosseinzadeh N & Assari A 2014 *Int.. J. Appl. Math. Res.* **3** 54–57
[22] Jog S R & Kotambari R 2016 *J. Math.* **2016** 1–11
[23] Kakeri F & Erfanian A 2015 *J. Prime Res. Math.* **11** 55–60
[24] Maas C 1985 *J. Comput. Appl. Math.* **12&13** 459–466
[25] Nawawi A & Rowley P 2015 *Electron. J. Combin.* **22** 1–21
[26] Philipose R S & Sarasija P B 2018 *Int. J. Eng. Tech.* **7** 1243– 1244
[27] Praba B & Obilia X A B 2018 *Int. J. Eng Tech.* **7** 1926–1930
[28] Stevanović D, Gutman I, & Rehman M U 2015 *Ars Math. Contemp.* **9** 109–113
[29] Vahidi J & Talebi A A 2010 *J. Math. Comput. Sci.* **1** 123–127
[30] Wang L & Wang Q 2010 *Discret. Math.* **310** 812–818
[31] Woodcock T 2015 *Int. J. Contemp. Math. Sci.* **10** 287–309
[32] Yin S 2008 *WSEAS Tran. Sys.* **7** 4 362–372