

PAPER • OPEN ACCESS

## Kruskal-Szekeres coordinates of spherically symmetric solutions in theories of gravity

To cite this article: A Romadani and M F Rosyid 2021 *J. Phys.: Conf. Ser.* **1816** 012030

View the [article online](#) for updates and enhancements.



The Electrochemical Society  
Advancing solid state & electrochemical science & technology

**240th ECS Meeting** ORLANDO, FL

Orange County Convention Center Oct 10-14, 2021



Abstract submission due: April 9

**SUBMIT NOW**

# Kruskal-Szekeres coordinates of spherically symmetric solutions in $f(R)$ theories of gravity

A Romadani<sup>1</sup> and M F Rosyid<sup>2</sup>

<sup>1</sup>Department of Physics, Faculty of Science and Technology, Universitas Islam Negeri Maulana Malik Ibrahim Malang, Malang, Indonesia

<sup>2</sup>Department of Physics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Yogyakarta, Indonesia

E-mail: arista.romadani@uin-malang.ac.id

**Abstract.** Understanding the missing matter problem in cosmological phenomena and scales of astrophysical is usually studied by modifying general relativity theory. In this article, we formulated the Kruskal-Szekeres coordinate of vacuum modified gravity model in  $f(R)$  theory. The generalization of the field equation was obtained by generalizing Hilbert-Einstein's action with gravitational Lagrangian in terms of  $f(R)$  function. By consider a special class of  $f(R)$  theory by taking  $R = R_0$ , we found the solution of static spherically symmetric spacetime that was known as de Sitter-Schwarzschild spacetime. The transformation rules were constructed from Kruskal-Szekeres coordinates in  $f(R)$  theory of modified general relativity to the Kruskal-Szekeres coordinate in general relativity theory. For  $\lambda \approx 0$ , the Schwarzschild and Kruskal-Szekeres metric for static spherically symmetric on  $f(R)$  theory reduced to the standard Schwarzschild and Kruskal-Szekeres metric on general relativity. We also show the spacetime structure of de Sitter-Schwarzschild and Kruskal-Szekeres coordinate. This work could open a promising way to understand some features of a black hole in the  $f(R)$  theory of gravity.

## 1. Introduction

General relativity theory has been able to explain the astronomical phenomena, covering the structure of massive objects like stars, including white dwarfs, neutron stars, black hole, quasar, and as a whole of the universe [1]. A massive star has a powerful gravitational field that light could not escape. The emitted light from the star's surface will be pulled back by gravity before it could be moving away from the star [2]. Scientists have found some weakness of Einstein's theory such as a cosmological constant that correlated with the existence of dark energy and inflation of universe that are unsolved problems recently. They began wondering whether the gravitational concepts on general relativity theory is a fundamental theory to describe the gravitational interaction [3].

The formulation of modified general relativity theory was proposed to construct the general relativity theory in the semi-classical scheme and the most fruitful approach was established by using  $f(R)$  theory [4, 5, 6, 7]. It has become a new paradigm in explaining the gravitational interaction [8].



Modified general relativity theories is constructed by two variational principles where it could be applied to the Einstein-Hilbert action: Palatini variation formalism and standard metric variation [9]. Both of these principles are built on the field equation with the Lagrangian is linear in  $R$  [10]. Brans and Dicke completed the extension of general relativity theory with scalar-tensor theories of gravity. It is one of the special cases where the gravitational interactions are associated with the scalar field as well as the tensor field in general relativity theory [11].

The fundamental principles of gravity theory were introduced by  $f(R)$  theory in the metric and Palatini formalism and scalar-tensor theory [12, 13]. The Axial symmetry solution could be solved using the Noether symmetry approach, starting from the spherical symmetry solutions in  $f(R)$  theory [14]. The idea is one of the most straightforward modifications of gravity theory with the generalization of the scalar Ricci of the Hilbert-Einstein equation to the function  $f(R)$  of  $R$  [15, 16, 17].

Kruskal-Szekeres is an explicit maximally extended and regular covering of the Schwarzschild spacetime [18, 19, 20]. Schwarzschild spacetime has provided an excellent physical description of the black hole's exterior region, but Kruskal-Szekeres described both the exterior and interior regions of a black hole [21]. Kruskal and Szekeres recognize that the singularity is unique but is not singular [22]. By transformation to Kruskal-Szekeres coordinate, we obtained the real singularity in a black hole.

The difference between observational evidence and calculation encourages some ideas based on the Einstein field equations. First, we showed dark matter by modelling the right side of Einstein's field equations. Second, we modified the Einstein field equation's left side by assuming that nothing additional material beyond the material appears. Then, starting from the idea, we modified the Einstein field equation's left side with  $f(R)$  gravity theory [23] and transformed the modified Schwarzschild black hole to the Kruskal-Szekeres coordinate.

$f(R)$  gravity theory is one of the most successful approaches resulted in which have become a paradigm in the study of the gravitational interaction. General relativity theory cannot work as a fundamental theory when a full quantum description of spacetime and gravity is sought for. Various alternative gravitational theories were proposed which attempt to formulate and it successfully could be replicated. Other reasons to modify general relativity are provided by the attempt to fully incorporate Mach's principle into the theory. General relativity contains only some of Mach's ideas and admits solutions that are explicitly anti-Machian, such as the Godel universe and exact pp-waves [24, 8].

In massive objects such as white dwarfs, neutron stars and black hole, the field equations represented strong gravitational interaction. In relativistic stars, it was related to the singularity problem of  $f(R)$  dark energy models in the high-curvature regime. Under the weak gravity backgrounds where the background metric is described by a Minkowski space-time, spherically symmetric solutions have been derived [25, 26, 27].

For relativists,  $f(R)$  theory seems more appealing because its more apparent geometrical nature, whereas the Brans-Dicke theory seems more appealing to particle physicists. Its theoretical implications are to restore the geometrical role and use it to define the covariant derivatives in the matter action. This is particularly useful to place constraints on inflation and dark energy models based on  $f(R)$  theories [28]. So in this article, we reviewed the alternative of general relativity by using  $f(R)$  theory because in both the metric and the Palatini formalisms it can acquire a Brans-Dicke theory representation. We reviewed  $f(R)$  for  $R$  is a constant because the spherically symmetric solution yield generalization Schwarzschild spacetime with a cosmological constants where it was related with unsolved problems currently in general relativity theory: dark energy model and inflation of universe.

## 2. Formalism

Einstein-Hilbert action equation is written by

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + S^{(m)} \quad (1)$$

variational principles of action in equation (1) yield the field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad (2)$$

known as the Einstein field equation [29]. Einstein field equations described the relation of matter distribution and spacetime curvature. The gap between observations and calculation results has arisen from dark energy and dark matter to modify the Einstein field equations. The right side modification of Einstein field equations related to the energy-momentum tensor according to the matter's existence. Meanwhile, the left side of Einstein field equations through  $f(R)$  theory associated with spacetime geometry presents a geometric model for dark energy.

### 2.1 Schwarzschild Black Hole

The metric of Schwarzschild spacetime as the solution of Einstein field equation for static and isotropic black hole masses  $m$ . The Schwarzschild metric from spherically symmetry solution is written for equation (3) [30]

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (3)$$

where singularity at  $r=0$  (real singularity) and  $r=2m$  (pseudo-singularity). The light cone of spacetime structure explained that an object or light that passes through pseudo-singularity (event horizon) would not be able to escape. Then, it will fall toward the singularity in  $r=0$ . An observer at rest in  $r > 2m$  will observe another observer toward the event horizon, and it is getting slower. Suppose the object or light is passing through from the event horizon for region  $r < 2m$ . In that case, the object or the light could not escape from the event horizon because it was forced and continued toward the singularity  $r=0$ .

### 2.2 Kruskal-Szekeres Coordinate

Schwarzschild spacetime is an exact spherically symmetric solution of Einstein equations. Kruskal-Szekeres coordinates was constructed where null geodesic (light cone) appeared as a straight line makes an angle  $45^\circ$  to the axis. The transformation from Schwarzschild to Kruskal-Szekeres coordinate conserved singularity at  $r=0$  and removed singularity at  $r=2m$ . So, the singularity at  $r=0$  is called as real singularity because it did not depend on the coordinate transformation.

Transformation from Schwarzschild black hole to the Kruskal-Szekeres coordinates for region  $r > 2m$  is [31]

$$T = \exp\left(\frac{r}{2R_s}\right) \left(1 - \frac{r}{R_s}\right)^{\frac{1}{2}} \sinh\left(\frac{t}{2R_s}\right) \quad (4)$$

$$X = \exp\left(\frac{r}{2R_s}\right) \left(1 - \frac{r}{R_s}\right)^{\frac{1}{2}} \cosh\left(\frac{t}{2R_s}\right) \quad (5)$$

and for  $r < 2m$  we obtained

$$T = \exp\left(\frac{r}{2R_s}\right) \left(1 - \frac{r}{R_s}\right)^{\frac{1}{2}} \cosh\left(\frac{t}{2R_s}\right) \quad (6)$$

$$X = \exp\left(\frac{r}{2R_s}\right) \left(1 - \frac{r}{R_s}\right)^{\frac{1}{2}} \sinh\left(\frac{t}{2R_s}\right) \quad (7)$$

with the metric of Kruskal-Szekeres spacetime is

$$ds^2 = -4 \left(\frac{R_s^3}{r}\right) \exp\left(-\frac{r}{R_s}\right) (dT^2 - dX^2). \quad (8)$$

### 3. Results and Discussions

Generalization of Einstein-Hilbert action that is

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S^{(m)} \quad (9)$$

variational principles of equation (7) is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{f'(R)} \left[ \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \nabla_\mu \nabla^\mu f'(R) + \frac{\{f(R) - f'(R)R\}}{2} g_{\mu\nu} \right] + T_{\mu\nu} \quad (10)$$

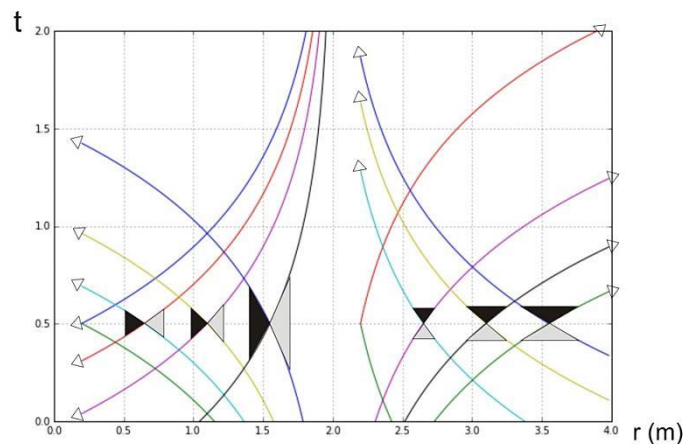
the solution of equation (10) for the constant curvature case known as black hole in de-Sitter spacetime, for  $R = 0$  is minkowski spacetime, and for  $f(R) = R$  we obtained the Einstein's field equation [19,20]. Modified Schwarzschild metric by using  $f(R)$  theory that is [32]

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (11)$$

equation (11) is a metric representation for spherically symmetric solution by using  $f(R)$

theory where  $\lambda$  is a cosmological constant satisfy  $\lambda = \frac{1}{4} R$  [17]. For  $\lambda \approx 0$ , it returns to the equation

(3) and we can say that equation (11) is generalization of the standard Schwarzschild metric. The spacetime structure of metric representation in equation (11) was shown in Figure 1 and it appeared that the light cone become different for region  $r > 2m$  and region  $r < 2m$ .



**Figure 1.** Modified Schwarzschild diagram showing the trajectory of an object or light in the Schwarzschild coordinate

Physically, Figure 1 has the same interpretation with the Schwarzschild spacetime, if an object or light passing through the event horizon ( $r = 2m$ ) then it will not be able to escape and then fall down toward singularity  $r = 0$ .

We constructed the transformation of modified Schwarzschild spacetime to the Kruskal-Szekeres coordinate. We introduce coordinate  $u$  and  $v$  with the relation

$$u = t + r \quad (12)$$

and

$$v = t - r \quad (13)$$

where interval of the coordinate  $-\infty < u/v < \infty$  and we get the relation

$$dudv = dt^2 - dr^2 \quad (14)$$

so, the metric of modified Schwarzschild spacetime in equation (11) is written by (2-dimensional case for  $\theta$  and  $\phi$  constant)

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right) dudv. \quad (15)$$

For  $r > 2m$ , we define the functions

$$U = \exp\left(\frac{u(4\lambda m^2 - 1)}{4m}\right) \quad (16)$$

and

$$V = -\exp\left(\frac{-v(4\lambda m^2 - 1)}{4m}\right) \quad (17)$$

from equation (16) and (17) we obtained the differential form

$$dU = \frac{(4\lambda m^2 - 1)}{4m} \exp\left(\frac{u(4\lambda m^2 - 1)}{4m}\right) du \quad (18)$$

and

$$dV = \frac{(4\lambda m^2 - 1)}{4m} \exp\left(\frac{-v(4\lambda m^2 - 1)}{4m}\right) dv \quad (19)$$

we get the relation that

$$dudv = \left(\frac{16m^2(r - 2m)}{(4\lambda m^2 - 1)^2}\right) dUdV \quad (20)$$

metric of modified Schwarzschild spacetime from equation (15)

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right) \left(\frac{16m^2(r - 2m)}{(4\lambda m^2 - 1)^2}\right) dUdV. \quad (21)$$

Now, we introduce the new coordinate  $T$  and  $X$  with relation

$$T = \frac{U + V}{2} \quad (22)$$

and

$$X = \frac{U - V}{2} \quad (23)$$

where interval  $-\infty < T/X < \infty$ , from equation (22) and (23) we know that

$$dUdV = dT^2 - dX^2 \quad (24)$$

metric of modified Schwarzschild spacetime in the new coordinate is written by

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right) \left(\frac{16m^2(r-2m)}{(4\lambda m^2 - 1)^2}\right) (dT^2 - dX^2) \quad (25)$$

the new coordinate is called by Kruskal-Szekeres coordinate with the transformation rules are given by

$$T = \frac{1}{\sqrt{r-2m}} \sinh\left(\frac{(4\lambda m^2 - 1)t}{4m}\right) \quad (26)$$

$$X = \frac{1}{\sqrt{r-2m}} \cosh\left(\frac{(4\lambda m^2 - 1)t}{4m}\right) \quad (27)$$

where satisfy

$$T^2 - X^2 = \frac{1}{(r-2m)} \quad (28)$$

or

$$\frac{T}{X} = \tanh\left(\frac{(4\lambda m^2 - 1)t}{4m}\right). \quad (29)$$

For  $r < 2m$  with the same way as before, the transformation rules are given by

$$T = \frac{1}{\sqrt{2m-r}} \cosh\left(\frac{(4\lambda m^2 - 1)t}{4m}\right) \quad (30)$$

$$X = \frac{1}{\sqrt{2m-r}} \sinh\left(\frac{(4\lambda m^2 - 1)t}{4m}\right) \quad (31)$$

where satisfy

$$T^2 - X^2 = \frac{1}{(2m-r)} \quad (32)$$

or

$$\frac{T}{X} = \tanh\left(\frac{(4\lambda m^2 - 1)t}{4m}\right) \quad (33)$$

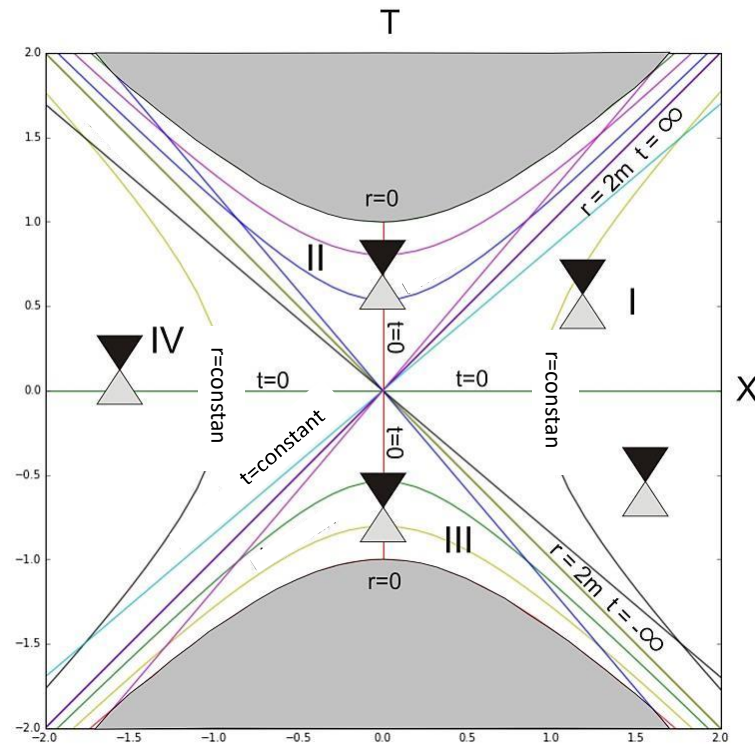
and the metric in the Kruskal-Szekeres coordinate is also given by equation (25).

The modified Schwarzschild spacetime in the Kruskal-Szekeres coordinates by using  $f(R)$  theory is shown in Figure 2. It appears that the light cone in the Figure 2 has  $45^\circ$  angle to the axis coordinate. Region I in Figure 2 is the representation for  $r > 2m$  of Schwarzschild spacetime. An object from region I towards  $r = 2m$  in  $X = \infty$  will appear through  $r = 2m$  in  $T = -\infty$  enter to region II. The

object will fall into singularity point at  $r = 0$  in  $T = -\sqrt{\frac{1}{2m} + X^2}$  and it will not be able to escape

from the area. Regions I and II are describe the interpretation of Schwarzschild spacetime known as black hole. Regions III and IV are duplicate of the regions I and II with time reversed characteristically, the phenomena called as white hole. White hole has the same physical description as black hole but it is a reverse processes. In region III, an object emerges from the singularity point at

$r = 0$  in  $T = \sqrt{\frac{1}{2m} + X^2}$  toward  $r = 2m$  in  $T = \infty$  and region IV has similar properties to the region I for  $r > 2m$ .



**Figure 2.** Modified Kruskal-Szekeres diagram showing the trajectory of an object or light in the Kruskal-Szekeres coordinate

#### 4. Conclusions

We have constructed the Kruskal-Szekeres coordinate in  $f(R)$  theory as the extension of Schwarzschild spacetime. We obtained the static spherically symmetric solution of Schwarzschild black hole in  $f(R)$  theory, and the transformation rules from the Schwarzschild to Kruskal-Szekeres. An observer who was in region I could not communicate with the other observer in the region IV because light signals are sent from region I to IV, it will go into a black hole, and the spacetime singularity destroys it. Region III and IV are an extension of the Schwarzschild solution that satisfied the Einstein field equations.  $f(R)$  theory. From the mathematical results, we also obtained that the spherically symmetric solution in  $f(R)$  theory are similar metric to de-Sitter spacetime. In future work, the physical correct description of Schwarzschild black hole in Kruskal-Szekeres coordinate can open a promising way to explain the phenomena both exterior and interior regions of a black hole.

#### Acknowledgments

The author thanks to Universitas Islam Negeri Maulana Malik Ibrahim Malang for supporting this publication.

#### References

- [1] Calzà M, Rinaldi M and Sebastiani L 2018 A special class of solutions in  $F(R)$ -gravity *Eur. Phys. J. C*
- [2] Faraoni V, Capozziello S, Capozziello S and Faraoni V 2011 Extended gravity: a primer, in *Beyond Einstein Gravity*
- [3] Wald R M 1974 Black hole in a uniform magnetic field *Phys. Rev. D*
- [4] Carroll S M, Duvvuri V, Trodden M and Turner M S 2004 Is cosmic speed-up due to new



- gravitational physics? *Phys. Rev. D - Part. Fields, Gravit. Cosmol*
- [5] Dvali G, Gabadadze G and Porrati M, 2000 4D gravity on a brane in 5D Minkowski space *Phys. Lett. Sect. B Nucl. Elem. Part. High-Energy Phys*
  - [6] De La Cruz-Dombriz Á and Dobado A 2006  $f(R)$  gravity without a cosmological constant *Phys. Rev. D - Part. Fields, Gravit. Cosmol*
  - [7] Cembranos J A R 2006 The Newtonian limit at intermediate energies *Phys. Rev. D - Part. Fields, Gravit. Cosmol*
  - [8] Faraoni V and Capozziello S 2011 *Beyond Einstein Gravity*
  - [9] Ferraris M, Francaviglia M and Reina C 1982 Variational formulation of general relativity from 1915 to 1925 “Palatini’s method” discovered by Einstein in 1925 *Gen. Relativ. Gravit*
  - [10] Buchdahl H A 1970 Non-Linear Lagrangians and Cosmological Theory *Mon. Not. R. Astron. Soc*
  - [11] Brans C and Dicke R H 1961 Mach’s principle and a relativistic theory of gravitation *Phys. Rev*
  - [12] Capozziello S and de Laurentis M 2011 Extended Theories of Gravity *Physics Reports*.
  - [13] Capozziello S, Stabile A and Troisi A 2010 Comparing scalar-tensor gravity and  $f(R)$ -gravity in the Newtonian limit *Phys. Lett. Sect. B Nucl. Elem. Part. High-Energy Phys*
  - [14] Capozziello S, De Laurentis M and Stabile A 2010 Axially symmetric solutions in  $f(R)$ -gravity *Class. Quantum Gravity*
  - [15] Yadav B K and Verma M M 2019 Dark matter as scalaron in  $f(R)$  gravity models *J. Cosmol. Astropart. Phys*
  - [16] Sotiriou T P and Faraoni V 2010  $f(R)$  theories of gravity *Rev. Mod. Phys*
  - [17] Iorio L, Ruggiero M L, Radicella N and Saridakis E N 2016 Constraining the Schwarzschild-de Sitter solution in models of modified gravity *Phys. Dark Universe*
  - [18] Lake K 2006 Maximally extended, explicit and regular coverings of the Schwarzschild-de Sitter vacua in arbitrary dimension *Class. Quantum Gravity*
  - [19] Kruskal M D 1960 Maximal extension of schwarzschild metric *Phys. Rev*
  - [20] Szekeres G and Szekeres P 2002 On the singularities of a Riemannian manifold, *General Relativity and Gravitation*
  - [21] Mitra A 2012 Kruskal Coordinates and Mass of Schwarzschild Black Holes: No Finite Mass Black Hole at All *Int. J. Astron. Astrophys*
  - [22] DiNunno B S and Matzner R A 2010 The volume inside a black hole *Gen. Relativ. Gravit*
  - [23] de Felice A and Tsujikawa S 2010  $f(R)$  theories *Living Reviews in Relativity*
  - [24] Gödel K 2000 Rotating Universes in General Relativity Theory *Gen. Relativ. Gravit*
  - [25] Kobayashi T and Maeda K I 2008 Relativistic stars in  $f(R)$  gravity, and absence thereof *Phys. Rev. D - Part. Fields, Gravit. Cosmol*
  - [26] Frolov A V 2008 Singularity problem with  $f(R)$  models for dark energy *Phys. Rev. Lett*
  - [27] De Felice A and Tsujikawa S 2010  $f(R)$  Theories *Living Rev. Relativ*
  - [28] Sotiriou T P 2009 6+1 lessons from  $f(R)$  gravity in *Journal of Physics: Conference Series*
  - [29] Romadani A 2015 Schwarzschild Black Hole On The Extended of General Relativity Theory (Universitas Gadjah mada)
  - [30] Schutz B 2009 *A First Course in General Relativity*
  - [31] Weinberg S and Dicke R H 1973 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity *Am. J. Phys.* **41**(4) 598–599
  - [32] Capozziello S, Frusciante N and Vernieri D 2012 New spherically symmetric solutions in  $f(R)$ -gravity by Noether symmetries *Gen. Relativ. Gravit*