Brownian Motion Around Black Hole

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ABSTRACT

Brownian motion theory is always challenging to describe diffusion phenomena around a black hole, with the main issue is how to extend the classical theory of Brownian motion to the general relativity framework. In this study, we extended the Brownian motion theory in a curved space-time come from a strong gravitational field on the Schwarzschild black hole. The Brownian motion theory in Schwarzschild space-time was derived by using the Fokker-Planck equation, and the stationary solution was analyzed by Ito, Stratonovich-Fisk, and Hanggi-Klimontovich Approach. The numerical result was found that the Brownian motion in Schwarzschild space-time $\chi \leq 1$ was reduced to the standard Brownian motion in Newtonian classical theory. According to the Hanggi-Klimontovich approach for $\chi \leq 1$ the result showed a consistent with the relativistic Maxwell distribution. The Fokker-Planck equation in Schwarzschild space-time was also formulated as a generalization of relativistic Brownian motion theory. This work could open a promising interpretation to formulate the diffusion phenomena around a massive object in the general relativity framework.

Keywords: *Brownian Motion, Diffusion Process, Schwarzschild, Fokker-Planck, General Relativity, Black Hole*

1. INTRODUCTION

In 1828, Robert Brown discovered the physical phenomena of Brownian motion, and he observed through a microscope the irregular motion of pollen grains in water. The movement is known as a random collision between the molecule and liquid, and now we understood to be due to molecular bombardment. In mathematics, it can be explained by the stochastic process [1],[2]

Bachelier developed the theory of Brownian motion to complete his thesis. Einstein explained the formulation problem of a Brownian motion theory in detail and accurately predicted the particle's random irregular motions [3],[4]. He also completed the theory of the diffusion process in the special relativity framework [5].

The important basic concepts of Brownian motion from a collision between the molecule and liquid in the fluid are viscosity friction and stochastic force proposed by [6]. They also generalize the theory of diffusion process in Minkowski space-time; it is known as the relativistic diffusion process [7],[8],[9].

The Langevin equation is a fundamental equation to describe the Ornstein-Uhlenbeck process; it contains both frictional forces and random forces. The three most popular interpretations solved the Langevin equation pioneered by Ito [10],[11],[12] with the pre-point discretization rule which is evaluated the boundary at the lower interval. Stratonovich-Fisk also proposed the rule to evaluate the boundary at the mid-point interval [13],[14], Hanggi-Klimontovich approach evaluated the boundary at the upper interval called by post-point rule [15],[16],[17].

The formulation of Brownian motion in a viscous medium for the (1+1)-dimensional case can be generalized for special relativity framework corresponding relativistic Langevin Equation [18]. It can also be expanded by adding the external field to the (1+3)-dimensional case and becoming distinctly by expanding from one to three spatial dimensions [19]. In special relativity, the diffusion process on velocity distribution consistent with the generalization of Maxwell distribution [20]. Furthermore, the relativistic diffusion process is also generalized to the general relativity framework in curved space-time [21],[22]. For example, the simple case of diffusion applied for the Friedmann-Robertson-Walker (FRW) space-time [23].

Fokker-Planck equation (FPEs) generally explained the evolution of particle. It is written as a differential equation form to describe the probability density corresponding to the differential equation of the stochastic process [24,25]. FPEs can be reduced from the Langevin equation as an alternative model to solve the non-Markovian relativistic diffusion theory [26]. FPEs were successfully applied in plasma physics, high energy physics, and astrophysics in different areas of physics.

In astrophysics, a black hole is an important consequence of the existing general relativity theory. It is described as a region with extremely gravitational effects in which all objects, including light, cannot escape from the horizon [27]. Moreover, the motion of particles around the Black Hole when they are falling towards the Black Hole's center is known by accretion disk phenomena [28]. It can be regarded as diffusion phenomena, which could only be explained by general relativity theory [29]. Franchi dan Le Jan proposed the relativistic motion of the diffusion process in which the acceleration is given by white noise [30]. They provide the theory of diffusion problems in the Schwarzschild and Kruskal-Szekeres manifolds to represent the Black Holes. From the Bunster-Carlip equation, Ropotenko has derived a diffusion equation of black hole on Kerr-Newman space-time, and its solution has a Gaussian distribution form [31]. He also found the entropy of the diffusion process as well as the Bekenstein-Hawking entropy [32].

The fundamental concept in Brownian motion to describe the diffusion process is a diffusion coefficient. A constant diffusion coefficient is related to white noise and a friction coefficient by the fluctuation-dissipation theorem [33]. A diffusion coefficient constant of the Langevin equation is approximately solved by the Schwarzschild distribution, and it can be used to describe the distribution of the velocity component [34].

Starting from the Brownian motion theory in the special relativity framework explained before, the present article aims to extend the relativistic diffusion process into the diffusion phenomena in general relativity theory under the external force field from a strong gravitational field around the black hole by using Langevin dan Fokker-Planck equation. We also determine the correlation between probability density and velocity in curved space-time from Schwarzschild Black Hole, and we compare the result by using the three approaches: Ito, Stratonovich-Fisk, and Hanggi-Klimontovich approach. It is very useful to illustrate how does the diffusion process around Black Hole works.

In this research, we provide the Brownian motion theory according to the Fokker-Planck equation around the Schwarzschild space-time. First, we construct the relativistic diffusion process through the Fokker-Planck equation in a special relativity framework. Second, we define the Lorentz factor in the general relativity

framework, i.e., Schwarzschild Black Hole space-time. Third, the stationary solution of the Fokker-Planck equation is solved by using the three approaches: Ito approach, Stratonovich-Fisk approach, and Hanggi-Klimontovich approach. And the last, numerical results of a particle diffuses around Schwarzschild space-time are presented according to the diffusion coefficient parameter.

2. BROWNIAN MOTION THEORY

In this section, we describe the Brownian motion equation in the special relativity framework by using the Fokker-Planck equation. Explicitly, FPEs are expressed as a continuity equation. In the momentum representation, it can be defined as [18]

$$\frac{\partial}{\partial t} f(t, p) + \frac{\partial}{\partial p} j(t, p) = 0 \qquad (1)$$

where $f(t, p)$ are probability density and probability current is expressed by $j(t, p)$. For stationary solution it can be analyzed by Ito, Stratonovich-Fisk, and Hanggi-Klimontovich approach satisfy condition $j(p) = 0$. Assume the Ansatz solution is defined as

$$f(p) = C\gamma^{-\alpha} \exp(-\chi\gamma) \qquad (2)$$

with $\chi = \dfrac{vm^2c^2}{D}$ is the dimensionless parameter represent the ratio of rest energy and thermal energy of a particle.

Ito approach evaluates the boundary at the lower interval $w^\mu(\tau)$ satisfy the condition

$$\gamma = \gamma(p(t)) \qquad (3)$$

the current in equation (7) can be obtained as

$$j_I(t, p) = -\left[vpf + D\frac{\partial}{\partial p}\gamma(p)f \right]. \qquad (4)$$

Stratonovich-Fisk approach evaluates the boundary at the mid-point interval satisfy the condition

$$\gamma = \gamma\left(\frac{p(t) + p(t+dt)}{2} \right) \qquad (5)$$

where

$$j_{SF}(t, p) = -\left[vpf + D\sqrt{\gamma(p)}\frac{\partial}{\partial p}\sqrt{\gamma(p)}f \right] \qquad (6)$$

and Hanggi-Klimontovich approach evaluated the boundary at the upper interval satisfy the condition

$$\gamma = \gamma(p(t+dt)) \tag{7}$$

where

$$j_{HK}(t,p) = -\left[vpf + D\gamma(p)\frac{\partial}{\partial p}f\right]. \tag{8}$$

3. BROWNIAN MOTION ON SCHWARZSCHILD SPACE-TIME

The expansion of the diffusion process on general relativity starts from constructing the Brownian motion theory in special relativity, and we generalize the diffusion process on the Schwarzschild Black Hole space-time. The Lorentz factor for the Schwarzschild space-time is required to obtain the solution of the Fokker-Planck equation.

3.1 Schwarzschild Space-time

We consider the Schwarzschild Black Hole to describe how the particle is moving around the Black Hole. The Schwarzschild metric equation for (1+1)-dimensional case (θ, ϕ constant) is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)c^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \tag{9}$$

where $m = \dfrac{GM}{c^2}$ and the metric tensor of Equation (9) can be written by

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \dfrac{2m}{r}\right) & 0 \\ 0 & \left(1 - \dfrac{2m}{r}\right)^{-1} \end{pmatrix}. \tag{10}$$

From defining the proper time τ in relativity theory, we obtain that mathematically

$$d\tau = \frac{1}{\gamma_{Sch}(r,v)} dt \tag{11}$$

the Lorentz factor in Schwarzschild space-time $\gamma_{Sch}(r,v)$ is expressed by

$$\gamma_{Sch}(r,v) = \left[\left(1 - \frac{2m}{r}\right) - \left(1 - \frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}} \tag{12}$$

in momentum representation, it is written by

$$\gamma_{Sch}(r,p) = \left[\left(1 - \frac{2m}{r}\right)^{-1} + \left(1 - \frac{2m}{r}\right)^{-2}\frac{p^2}{m^2c^2}\right]^{\frac{1}{2}}. \tag{13}$$

3.2 Fokker-Planck Equation on Schwarzschild Space-time

We solve FPEs for a stationary solution $f(p) = 0$ with the three interpretations commonly used, namely Ito approach, Stratonovich-Fisk approach, and Hanggi-Klimontovich approach. In this approach, we evaluate the time interval boundary during $[t, t+dt]$.

We know the Lorentz factor in momentum representation is given by Equation (13), and we obtained useful mathematical tools as

$$\frac{\partial \gamma_{Sch}}{\partial p} = \frac{p}{\gamma_{Sch}\left(1 - \dfrac{2m}{r}\right)^2 m^2 c^2} \tag{14}$$

where $p = \gamma_{Sch}mv$, and

$$\left|\frac{\partial p}{\partial v}\right| = \gamma_{Sch}^3 m\left(1 - \frac{2m}{r}\right) \tag{15}$$

from Equation (2), we generalize the solution for the Schwarzschild space-time

$$\frac{\partial f}{\partial p} = \frac{-p}{\gamma_{Sch}m^2c^2}\left(\frac{\alpha}{\gamma_{Sch}} + \chi\right)f \tag{16}$$

the relation between the solution of FPEs with the probability density is defined by

$$\phi(v) = \left|\frac{\partial p}{\partial v}\right| f(p(v)). \tag{17}$$

2.2.1. Ito Approach

In the Ito approach, we discuss Ito interpretation of the multiplication rule of white noise in the stochastic differential equation. In this approach, we evaluate the boundary at pre-point discretization of the interval $[t, t+dt]$ satisfy

$$\gamma_{Sch} = \gamma_{Sch}(p(t)) \tag{18}$$

and the expression of Ito's current for a stationary condition is defined by Ito

$$vpf + D\frac{\partial}{\partial p}\gamma_{Sch}(p)f = 0$$

$$\left[v - \frac{D}{m^2 c^2}\left(\frac{\alpha}{\gamma_{Sch}} + \chi\right) + \frac{D}{\gamma_{Sch}\left(1-\frac{2m}{r}\right)^2 m^2 c^2} \right] pf = 0$$

(19)

which is fulfilled for $\alpha = \dfrac{1}{\left(1-\frac{2m}{r}\right)^2}$ and

$\chi = \dfrac{vm^2c^2}{D}$. From Equation (2) and (19), the solution of stationary FPEs is written by

$$f(v) = C_I\left(\left[\left(1-\frac{2m}{r}\right) - \left(1-\frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}\right)^{-\frac{1}{\left(1-\frac{2m}{r}\right)^2}} \times$$

$$\exp\left\{-\chi\left[\left(1-\frac{2m}{r}\right) - \left(1-\frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}\right\}$$

(20)

and from equation (17) and (20), the probability density of a particle around Schwarzschild space-time according to Ito's approach is given by

$$\phi_I(v) = C_I m\left(1-\frac{2m}{r}\right)\left(\left[\left(1-\frac{2m}{r}\right) - \left(1-\frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}\right)^{3-\frac{1}{\left(1-\frac{2m}{r}\right)^2}} \times$$

$$\exp\left\{-\chi\left[\left(1-\frac{2m}{r}\right) - \left(1-\frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}\right\}.$$

(21)

2.2.2. Stratonovich-Fisk Approach

In this approach, we evaluate the boundary at the midpoint of the interval $[t, t+dt]$

$$\gamma_{Sch} = \gamma_{Sch}\left(\frac{p(t) + p(t+dt)}{2}\right)$$

(22)

the current for a stationary condition is defined by Stratonovich-Fisk

$$vpf + D\sqrt{\gamma_{Sch}(p)}\frac{\partial}{\partial p}\sqrt{\gamma_{Sch}(p)}f = 0$$

$$\left[v - \frac{D}{m^2 c^2}\left(\frac{\alpha}{\gamma_{Sch}} + \chi\right) + \frac{D}{2\gamma_{Sch}\left(1-\frac{2m}{r}\right)^2 m^2 c^2} \right] pf = 0$$

(23)

which is fulfilled for $\alpha = \dfrac{1}{2\left(1-\frac{2m}{r}\right)^2}$ and

$\chi = \dfrac{vm^2c^2}{D}$. The solution of stationary FPEs is written by

$$f(v) = C_{SF}\left(\left[\left(1-\frac{2m}{r}\right) - \left(1-\frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}\right)^{-\frac{1}{2\left(1-\frac{2m}{r}\right)^2}} \times$$

$$\exp\left\{-\chi\left[\left(1-\frac{2m}{r}\right) - \left(1-\frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}\right\}$$

(24)

the probability density of a particle around Schwarzschild space-time according to the Stratonovich-Fisk approach is given by

$$\phi_{SF}(v) = C_{SF} m\left(1-\frac{2m}{r}\right)\left(\left[\left(1-\frac{2m}{r}\right) - \left(1-\frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}\right)^{3-\frac{1}{2\left(1-\frac{2m}{r}\right)^2}} \times$$

$$\exp\left\{-\chi\left[\left(1-\frac{2m}{r}\right) - \left(1-\frac{2m}{r}\right)^{-1}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}\right\}.$$

(25)

2.2.3. Hanggi-Klimontovich Approach

In the Hanggi-Klimontovich approach, we discuss the multiplication rule to evaluate the interval boundary $[t, t+dt]$ at the upper satisfy

$$\gamma_{Sch} = \gamma_{Sch}\left(p(t+dt)\right)$$

(26)

the current for a stationary condition is defined by Hanggi-Klimontovich

$$vpf + D\gamma_{Sch}(p)\frac{\partial}{\partial p}f = 0$$

$$\left[v - \frac{D}{m^2 c^2}\left(\frac{\alpha}{\gamma_{Sch}} + \chi\right) \right] pf = 0$$

(27)

which is fulfilled for $\alpha = 0$ and $\chi = \dfrac{v m^2 c^2}{D}$. The solution of stationary FPEs is written as

$$f(v) = C_{HK} \exp\left\{ -\chi \left[\left(1 - \frac{2m}{r}\right) - \left(1 - \frac{2m}{r}\right)^{-1} \frac{v^2}{c^2} \right]^{-\frac{1}{2}} \right\}$$

(28)

and the probability density of a particle around Schwarzschild space-time according to the Hanggi-Klimontovich approach is given by

$$\phi_{HK}(v) = C_{HK} m \left(1 - \frac{2m}{r}\right) \left[\left(1 - \frac{2m}{r}\right) - \left(1 - \frac{2m}{r}\right)^{-1} \frac{v^2}{c^2} \right]^{-\frac{3}{2}} \times$$

$$\exp\left\{ -\chi \left[\left(1 - \frac{2m}{r}\right) - \left(1 - \frac{2m}{r}\right)^{-1} \frac{v^2}{c^2} \right]^{-\frac{1}{2}} \right\}$$

(29)

for $r >> 2m$ in Equation (28), it reduces to the distribution function in special relativity framework, and it is known as Maxwell-Juttner distribution (relativistic Maxwell distribution) according to a small value of χ [20,35].

4. DISCUSSION

We simulate the diffusion process around Schwarzschild space-time to describe the Fokker-Planck equation. Here, we assume $c = m = 1$ and $v_0 = 0.4c$ during the time interval. From equation (21), (25), and (29), the probability density functions $\phi(v)$ have been plotted for stationary Fokker-Planck equation in Schwarzschild space-time. In Figures 1, 2, and 3, we show probability density function under the velocity $v(c)$ for different values of the parameter χ by using Ito (green color), Stratonovich-Fisk (blue color), and Hanggi-Klimontovich approach (red color). For a large value, $\chi = 10$, it is clearly shown that the probability density function has been a similar pattern with Gaussian distribution, and we can easily say that $mc^2 >> k_B T$ corresponding to small temperature values. The probability density function also decreases exponentially for a high velocity. Otherwise, a small value of χ (for $\chi \leq 1$) the probability density function deviated from Gaussian distribution, and its corresponding to the high-temperature values of $mc^2 << k_B T$. These results have confirmed that for a large of χ (low-temperature limit case), the distribution function of Brownian motion in Schwarzschild space-time is compatible with the standard Brownian motion in Newtonian classical theory. For a small value of χ (high-temperature limit case), the

distribution function on the Hanggi-Klimontovich approach was reduced and compatible with the relativistic Maxwell distribution [18].
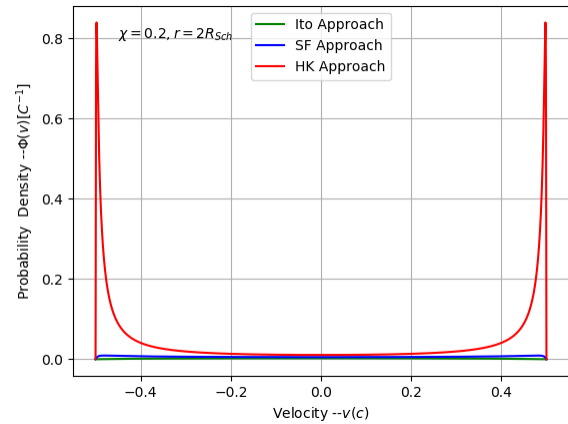

Figure 1 Stationary Fokker-Planck Equation by Ito, Stratonovich-Fisk and Hanggi-Klimontovich Approach for $\chi = 0.2$
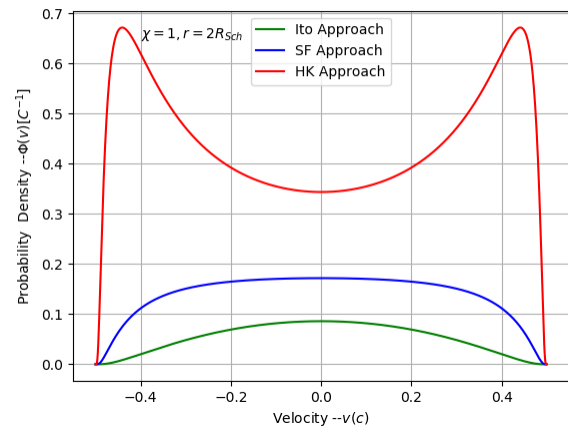

Figure 2 Stationary Fokker-Planck Equation for $\chi = 1$ approach by Ito, Stratonovich-Fisk, and Hanggi-Klimontovich.
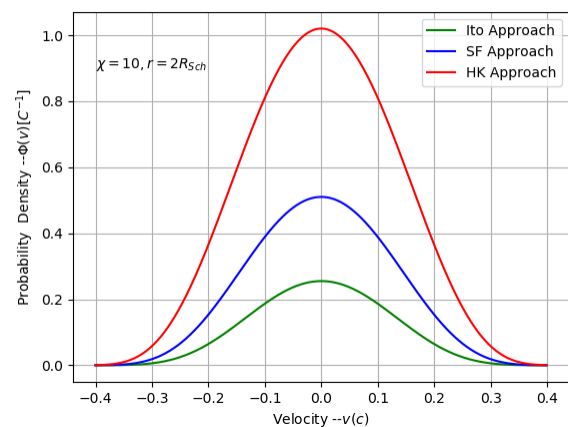

Figure 3 Stationary Fokker-Planck Equation by Ito, Stratonovich-Fisk and Hanggi-Klimontovich Approach for $\chi = 10$

4. CONCLUSION

We have generalized the relativistic diffusion theory for a Brownian particle to the curved space-time on general relativity using the relativistic Fokker-Planck. The Fokker-Planck equation for a stationary solution can be identified properly by the Ito approach, Stratonovich-Fisk approach, and Hanggi-Klimontovoich approach. The Newtonian limit of the probability density function for a large value χ has reduced to the standard form of Brownian theory. A small value χ on the Hanggi-Klimontovich approach has coincided with relativistic Maxwell distribution in the relativistic limit. The implication of the results is only the Hanggi-Klimontovich approach could have opted as an appropriate interpretation of the correct physical description. In future work, another possible interpretation will be formulated mathematically as an alternative interpretation, and the exact solution of the time-dependent Fokker-Planck (non-stationary problem) can be expressed with the correct physical description. From the results, we can develop several aspects of Brownian motion theory on the general relativity framework to the massive object such as in black hole, neutron star, and the physical correct description may solve the problem in the expanding universe.

AUTHORS' CONTRIBUTIONS

The author conceived and constructed the study. We extended the classical Brownian motion theory to the Brownian motion theory in a curved space-time in Schwarzschild black hole on general relativity theory.

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